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Applied Optics

VOLUME II

Applied Optics

The Computation of Optical Systems

Being the
"HANDBUCH DER ANGEWANDTEN
OPTIK" OF DR. ADOLPH STEINHEIL
AND DR. ERNEST VOIT

Translated & Edited by
JAMES WEIR FRENCH, B.Sc.

VOLUME II

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INTRODUCTORY

The following Section on "Symbols and Sign Convention" is repeated from Volume I, and the tables of formulæ which appeared throughout Volume I have been reproduced and extended to cover the general case of $t + 1$ *refracting surfaces.*

Symbols and Sign Convention

General Remarks

The point of intersection of the first refracting surface with the axis is the origin from which the reckoning is commenced.

Media are denoted by odd numbers; surfaces of separation are denoted by even numbers. Thus, for example, in the case of a simple lens, the medium from which the ray of light comes is denoted by the suffix -1, the first surface of separation has the suffix 0, the substance of the lens has the suffix $+1$, the second surface of separation has the suffix $+2$, and the medium into which the ray passes has the suffix $+3$. (Fig. 1.)

Symbols for Constants

Constants are indicated by small italic type. Thus n, n', n'', . . . represent the indexes of refraction of the various glasses used. For the glass of weakest refraction the letter n is used, and for the next stronger the letter n', and so on.

The numerical suffixes at the right lower side of the letters indicate the medium to which the index refers; the alphabetical suffixes at the left lower side indicate the corresponding Fraunhofer line. For example:—

$_{\mathrm{D}}n_1$ is the refractive index corresponding with the D line for a first crown-glass lens, and

$_{\mathrm{D}}n_3'$ is the refractive index, also for the D line, of a succeeding flint-glass lens separated by an interval from the crown-glass lens.

l, l', l'' are abbreviations for $\frac{1}{n-1}$, $\frac{1}{n'-1}$, and $\frac{1}{n''-1}$.

The suffixes corresponding with the media and colours are similar to those used in the case of n, n', n''.

Δn, $\Delta n'$, $\Delta n''$, . . . denote dispersions.

The suffixes for the media and colours are as before; for example:—

$\Delta_{\mathrm{D,F}}n_1$ is the value of the dispersion for a first crown-glass lens corresponding with a portion of the spectrum from D to F, and

$\Delta_{\mathrm{D,F}}n_3'$ is the value of the dispersion for the same portion of the spectrum in the case of an adjacent flint-glass lens.

s or t denotes a chosen term of a series.

$2s$ or $2t$ represent even terms, and $2s-1$ or $2t-1$ odd terms.

m denotes the value of the quotient: $\frac{\text{size of image}}{\text{size of object}}$.

When $m > 1$ the object appears magnified.
When $m < 1$ the object appears minified.
When $m = 1$ the object appears of natural size.

Symbols for the Positions of Points

The positions of points are indicated by capital letters.

The following points, A and A′, are distinguished by the suffixes of the corresponding refracting surface:—

A denotes the point in which an *incident* ray intersects the axis (fig. 2).

A′ denotes the point in which a *refracted* ray intersects the axis (fig. 2).

The following five letters are distinguished by suffixes, separated by a comma, indicating the number of the first and last refracting surfaces of the system:—

O denotes an object point for a system of refracting surfaces (fig. 3).

I denotes the corresponding image point (fig. 3).

E and E′ denote the first and second principal points (fig. 4; also fig. 13, Vol. I, p. 19).

F and F′ denote the first and second focal points (fig. 4; also figs. 11 and 12, Vol. I, p. 19).

N and N′ denote the first and second nodal points (fig. 15, Vol. I, p. 20).

The following points are distinguished by suffixes indicating the number of the corresponding refracting surface:—

C denotes the centre of the sphere of which the refracting surface is a portion (fig. 2).

V denotes the vertex of the refracting surface, that is to say, its point of intersection with the axis (fig. 2).

P denotes the point of intersection of an incident ray with the refracting surface (fig. 2).

Q denotes the point of intersection of an *incident* ray, or its prolongation, with the plane drawn normal to the axis through the centre of the surface (*centre plane*), (fig. 8).

Q′ denotes the point of intersection of a *refracted* ray, or its prolongation, with the corresponding centre plane (fig. 8).

R denotes the point of intersection of a line drawn through the point Q, in a direction opposite to that of the incident light and parallel to the axis, with a sphere having Q as centre and any chosen radius. (See Appendix II.)

R′ denotes the point of intersection of a line drawn through the point Q′, in a direction opposite to that of the incident light and parallel to the axis, with a sphere having Q′ as centre and any chosen radius. (See Appendix II.)

S denotes the point of intersection of an *incident* ray, or its prolongation, with the above-mentioned sphere. (See definition of R.)

S′ denotes the point of intersection of a *refracted* ray, or its prolongation, with the above-mentioned sphere. (See definition of R′.)

T denotes the point in the centre plane in which the prolongation of the line CQ intersects the chosen sphere. (See definition of R.)

T′ denotes the point in the centre plane in which the prolongation of the line CQ′ intersects the chosen sphere. (See definition of R′.)

The two following letters may have suffixes 1, 2, . . . to distinguish successive image planes:—

B denotes the point of intersection of the optical axis with a chosen image plane normal to the axis (fig. 8).

Y denotes the point of intersection of a refracted ray with the image plane drawn normal to the axis (fig. 8).

Symbols for Lengths

Lengths are indicated by small italic letters.

The two following lengths are measured from the vertex of the refracting surface whose number is appended as a suffix:—

a denotes the abscissa VA of the point of intersection of an *incident* ray with the axis (fig. 2).

a' denotes the abscissa VA′ of the point of intersection of a *refracted* ray with the axis (fig. 2).

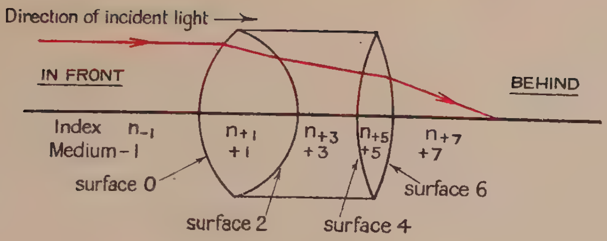


Fig. 1

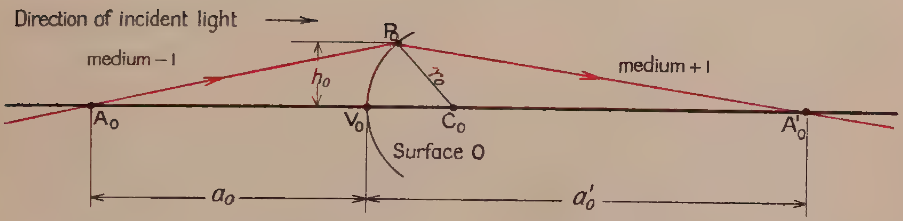


Fig. 2

a_0 negative
a'_0 positive

h_0 positive
r_0 positive

Abscissæ measured from V_0

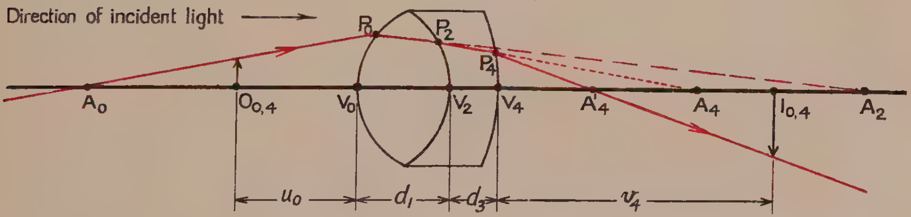
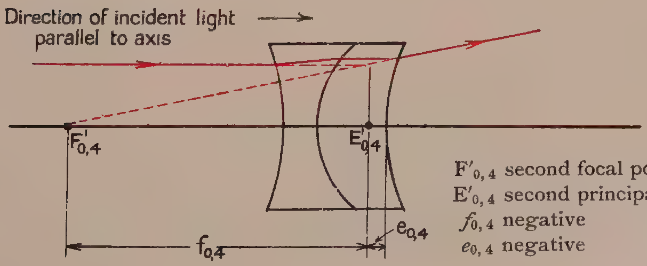


Fig. 3

a_0 negative
a_2 positive
a_4 positive

a'_4 positive
r_0 positive
r_2 negative

r_4 negative
u_0 negative
v_4 positive



$F'_{0,4}$ second focal point
$E'_{0,4}$ second principal point
$f_{0,4}$ negative
$e_{0,4}$ negative

Fig. 4

[*Facing p. 4.*

The three following letters are distinguished by the suffix of the corresponding refracting surface:—

u denotes the object distance, that is, the distance of an object point lying in the axis from the *first* refracting surface (fig. 3).

v denotes the image distance, that is, the distance from the *last* refracting surface of an image point on the axis (fig. 3).

v' denotes the distance of the best focus plane from the last refracting surface.

The two following letters are distinguished by suffixes corresponding with the medium between the vertexes of the refracting surfaces:—

c denotes generally the separation of the centre points of two consecutive refracting surfaces.

d denotes the separation of the vertexes of two consecutive refracting surfaces, that is, the thickness, which is always positive (fig. 3).

$\Delta d = 2r \sin^2 \frac{\eta}{2}$. This versine of the semi-aperture angle of a surface is distinguished by a suffix corresponding with the surface in question.

The three following letters are distinguished by suffixes separated by a comma, corresponding with the first and last refracting surfaces:—

e denotes the distance of the principal point from the vertex of the last refracting surface (fig. 4).

f denotes the true focal length EF of a system of refracting surfaces (fig. 4).

g denotes the distance of the nodal point from the vertex of the last refracting surface.

The five following letters are distinguished by suffixes indicating the corresponding refracting surfaces:—

h denotes the height above the axis of the point of intersection of a ray with a refracting surface (fig. 2).

h' denotes the height above the axis of the point of intersection of a ray with the focal plane.

r is the radius of a refracting surface (fig. 2).

q is the radius vector CQ in the centre plane (fig. 8).

q' is the radius vector CQ′ in the centre plane (fig. 8).

u^* denotes the height of an object.

v^* denotes the height of an image.

The three following letters are distinguished by suffixes indicating the successive image planes:—

b denotes the distance VB of a chosen image plane, normal to the axis, from the last refracting surface.

x and y are the rectangular co-ordinates of a point Y in which an emergent ray intersects the chosen normal image plane. The value of x is measured upwards from the point B, and the value of y is measured towards the side (fig. 8).

$z = \sqrt{x^2 + y^2}$ denotes the radius vector BY in the image plane normal to the axis (fig. 8).

Symbols for Angles

Angles are indicated by small Greek letters, and are distinguished by the suffixes of the corresponding refracting surfaces.

α denotes the angle which an *incident* ray in an axial plane makes with the axis (fig. 5).

α' denotes the angle which a *refracted* ray in an axial plane makes with the axis (fig. 5).

β denotes the semi angular aperture for a ray parallel to the axis and incident upon a system of refracting surfaces. It represents the angle which the ray incident parallel to the axis makes with the axis after each refraction. This semi angular aperture has suffixes separated by a comma, indicating the first and last refracting surfaces. The angle β is a special case of α', when $\alpha = 0$ (fig. 7).

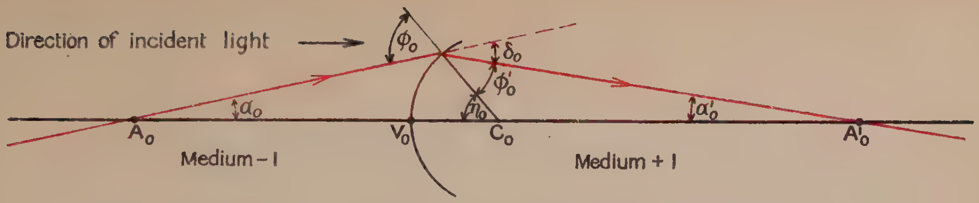


Fig. 5 α_0 negative, α'_0 positive, δ_0 positive, η_0 positive

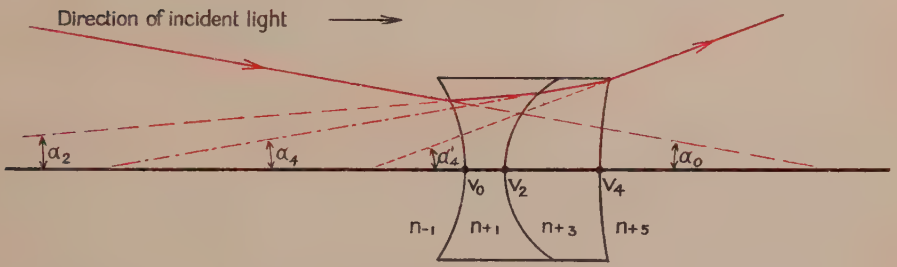


Fig. 6 α_0 positive, $\alpha_2 = \alpha'_0$ negative, $\alpha_4 = \alpha'_2$ negative, α'_4 negative

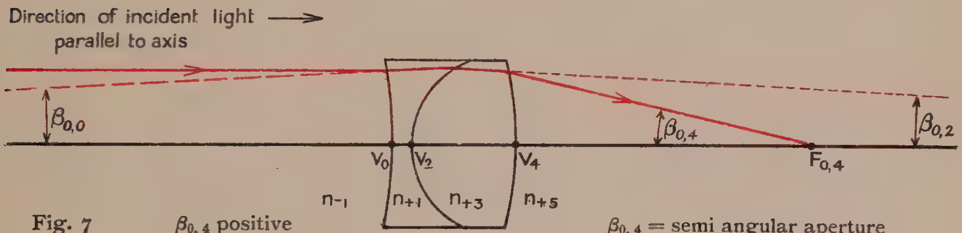


Fig. 7

$\beta_{0,4}$ positive
$\beta_{0,2}$ positive
$\beta_{0,0}$ negative

$\beta_{0,4}$ = semi angular aperture
of whole system

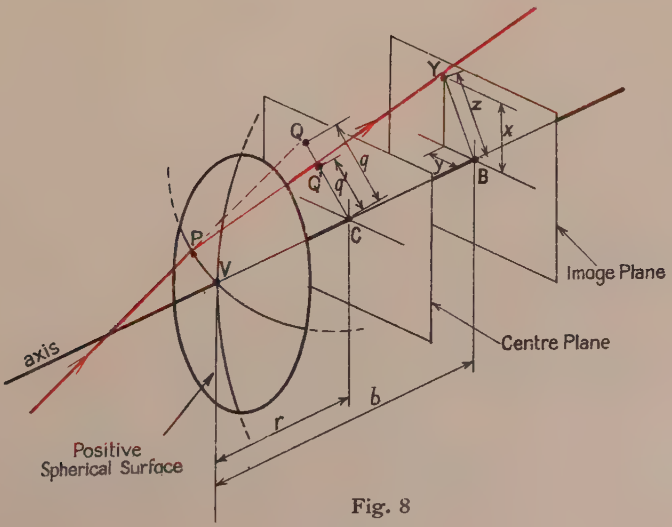


Fig. 8

ϵ denotes the semi angular aperture of an incident ray parallel to the principal ray and lying out of the plane containing the axis. It is the angle which the ray originally incident parallel to the axis makes after each refraction with the direction of the incident principal ray. The angle ϵ for rim rays out of the axial plane corresponds with the angle β for rim rays of an incident beam that is parallel to the axis (fig. 9).

ϕ denotes the angle of incidence (fig. 5).

ϕ' denotes the angle of refraction (fig. 5).

$\delta = (\phi - \phi')$ denotes the angle of deviation (fig. 5).

γ denotes the total angle of deviation of an incident ray by a prism.

ψ denotes the angle of a test prism.

ϑ denotes the angle of a second test prism to be combined with the first.

χ denotes the angle of reflection.

η denotes the angle which the radius makes with the optical axis (fig. 5).

λ denotes the angle which an *incident* ray in the plane of incidence makes with the line CQ (the internal angle at Q in the triangle PQC) (fig. 10).

λ' denotes the angle which a *refracted* ray in the plane of incidence makes with the line CQ′ (the internal angle at Q′ in the triangle PQ′C) (fig. 10).

τ denotes the angle which an *incident* ray makes with the line drawn parallel to the axis through Q (fig. 10).

τ' denotes the angle which a *refracted* ray makes with the line drawn through Q′ parallel to the axis (fig. 10).

The following four angles are reckoned from a vertical line drawn upwards, and are measured from 0° to 360° in a clockwise direction when viewed against the direction of the incident light:—

π denotes the angle which the projection of an *incident* ray on the centre plane makes with the vertical drawn upwards at the point Q (fig. 10).

π' denotes the angle which the projection of a *refracted* ray on the centre plane makes with the vertical drawn upwards at the point Q′ (fig. 10).

ζ denotes the angle which the polar co-ordinates CQ and CQ′ in the centre plane make with the vertical drawn upwards through the centre C (fig. 10).

ξ denotes the angle which the polar co-ordinate BY in a chosen image plane makes with the vertical drawn upwards through the point B. This angle may be distinguished by suffixes 1, 2 . . ., indicating the image planes (fig. 8).

ω denotes the angle such that $\tan \omega = \frac{n_{s-1}}{n_{s+1}}$. This angle is distinguished by the suffix indicating the corresponding refracting surface.

Sign Convention

Lengths

The direction in which the incident light travels is reckoned as positive, and the opposite direction as negative (fig. 1).

Lengths measured along the axis or parallel to it are positive when measured from the vertex of the refracting surface in the direction of the incident light; they are negative when measured in the opposite direction (fig. 4).

A focal length is positive when, moving in the direction of the incident light, the principal point precedes the focal point, and negative in the reverse case (fig. 4).

Lengths perpendicular to the axis are positive when they lie above the axis, and negative when below (fig. 2).

A radius vector is always positive.

Radii of curvature may be positive or negative. The radius of a surface that presents its convex face towards the incident light is regarded as being positive, and that of a surface which is concave to the incident light as negative (fig. 2).

Thicknesses are regarded as being always positive.

Thus in fig. 2 the distance a_0 is negative, being measured

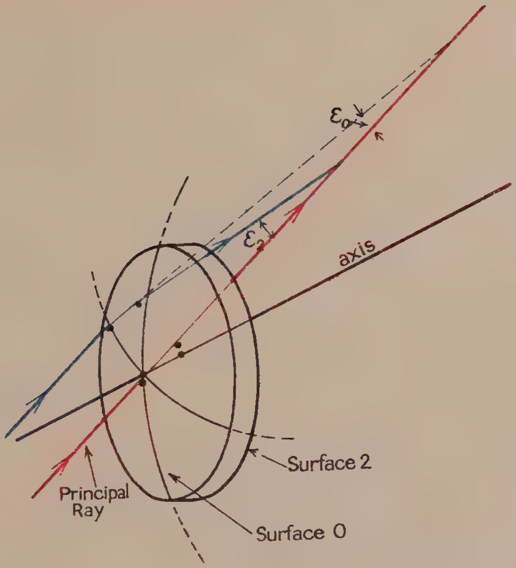


Fig. 9

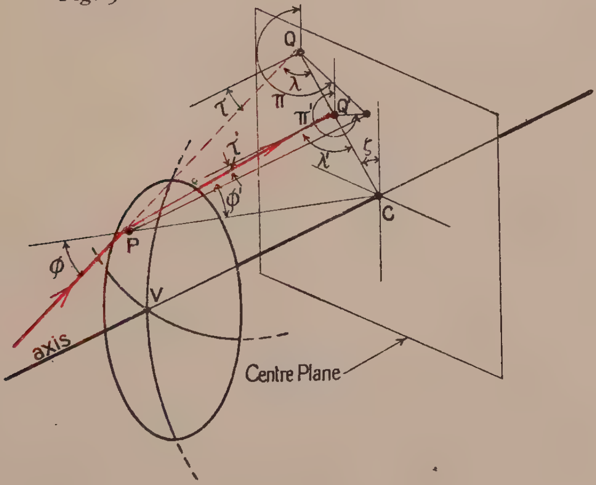


Fig. 10

[*Facing p. 8.*

from V_0 against the direction of the incident light, and a'_0 is positive, being measured in the direction of the incident light. The radius r_0 is positive, as the convex face is directed against the incident light. The incidence height h_0 is positive, since it lies above the axis.

In fig. 3, a_0 and u_0 are negative, being measured from their respective surfaces in the direction against that of the incident light. r_2 and r_4 are negative, since the concave surfaces are opposed to the direction of the incident light. The thicknesses, such as d_1 and d_3, are always numerically positive.

In fig. 4 the focal length $f_{0,4}$ is negative, since it precedes the principal point $E'_{0,4}$, and similarly $e_{0,4}$ is negative, since the principal point precedes the last vertex.

Angles

The semi angular aperture β is measured from the incident rim ray parallel to the axis to the refracted ray produced backwards. It is regarded as positive when, the eye being placed at their point of intersection, and looking against the direction of the incident light, the refracted ray appears above the incident ray, as in fig. 19, Vol. I, p. 28, and negative when it appears below, as in fig. 20 on same page.

All angles of deviation are measured from the incident ray, produced if necessary, to the refracted ray. The angle of deviation is reckoned as positive when the ray on refraction is bent towards the axis, and negative when bent from the axis.

The angle which a ray makes with the axis is positive when, the eye being placed at the point of intersection of the ray and the axis and looking against the direction of the incident light, the ray appears above the axis, and negative when below.

Thus in fig. 5 the angle α_0 is negative, because, when the eye is placed at A_0 and looks against the direction of the incident light, the ray appears to lie below the axis. The angle α'_0 is positive, because the eye, placed at A'_0 and looking against the incident light, sees the ray above the axis.

The signs of ϕ, ϕ', δ, γ, ψ, ϑ, χ, and η are determined by the signs of the lengths and angles from which they are derived. Thus in fig. 5 the angle η_0 is positive, since the elements which determine it—namely, the radius r_0 and the incidence height h_0—are both positive.

In fig. 6, p. 6, α_0 is positive because the ray appears above the axis. α_2, α_4, and α'_4 are all negative, because the rays when extended appear to the suitably-placed eye to lie below the axis.

In fig. 7 the semi angular apertures $\beta_{0,4}$ and $\beta_{0,2}$ are both positive, while $\beta_{0,0}$ is negative, in accordance with the rule.

In fig. 9, p. 8, the semi angular apertures ϵ_0 and ϵ_2, in the case of an oblique ray, are both positive under the conditions illustrated.

FORMULÆ FOR THE TRIGONOMETRICAL COMPUTATION OF RAYS THROUGH A LENS SYSTEM OF $(t + 1)$ REFRACTING SURFACES

I. FOR RAYS PROCEEDING FROM AN INFINITELY DISTANT AXIAL OBJECT POINT

(*a*) No Radius very large.

Paraxial Ray.	Rim Ray.
$\phi_0 = \frac{h_0}{r_0}.$	$\sin\phi_0 = \frac{h_0}{r_0}.$
$\phi_0' = \phi_0 \frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0 \frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$	$\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$
$a_0' - r_0 = \frac{r_0 \phi_0'}{\beta_{0,0}}.$	$a_0' - r_0 = \frac{r_0 \sin\phi_0'}{\sin\beta_{0,0}}.$
$\phi_2 = \frac{(a_0' - r_2 - d_1)\beta_{0,0}}{r_2}.$	$\sin\phi_2 = \frac{(a_0' - r_2 - d_1)\sin\beta_{0,0}}{r_2}.$
$\phi_2' = \phi_2 \frac{n_1}{n_3}.$	$\sin\phi_2' = \sin\phi_2 \frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\beta_{0,2} = \delta_2 + \beta_{0,0}.$	$\beta_{0,2} = \delta_2 + \beta_{0,0}.$
$h_2 = r_2(\phi_2' + \beta_{0,2}).$	$h_2 = r_2 \sin(\phi_2' + \beta_{0,2}).$
$a_2' - r_2 = \frac{r_2 \phi_2'}{\beta_{0,2}}.$	$a_2' - r_2 = \frac{r_2 \sin\phi_2'}{\sin\beta_{0,2}}.$
. . . .	. . . .
$\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\beta_{0,2t-2}}{r_{2t}}.$	$\sin\phi_{2t} =$ $= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\beta_{0,2t-2}}{r_{2t}}.$
$\phi'_{2t} = \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$	$\sin\phi'_{2t} = \sin\phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$

(*a*) No Radius very large—*Cont.*

Paraxial Ray.	Rim Ray.
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$
$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$	$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$
$h_{2t} = r_{2t}(\phi'_{2t} + \beta_{0,2t}).$	$h_{2t} = r_{2t} \sin(\phi'_{2t} + \beta_{0,2t}).$
$a'_{2t} - r_{2t} = \dfrac{r_{2t}\phi'_{2t}}{\beta_{0,2t}}.$	$a'_{2t} - r_{2t} = \dfrac{r_{2t} \sin \phi'_{2t}}{\sin \beta_{0,2t}}.$
$v_{2t} = a'_{2t}.$	
$f_{0,2t} = \dfrac{h_0}{\beta_{0,2t}}.$	$f_{0,2t} = \dfrac{h_0}{\tan \beta_{0,2t}}.$
$e_{0,2t} = a'_{2t} - f_{0,2t}.$	$e_{0,2t} = a'_{2t} - f_{0,2t}.$

FOR AN INFINITELY DISTANT AXIAL OBJECT POINT

(*b*) The first Radius infinite, the others not very large.

Paraxial Ray.	Rim Ray.
$\phi_2 = \dfrac{h_0}{r_2}.$	$\sin \phi_2 = \dfrac{h_0}{r_2}.$
$\phi'_2 = \phi_2 \dfrac{n_1}{n_3}.$	$\sin \phi'_2 = \sin \phi_2 \dfrac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi'_2.$	$\delta_2 = \phi_2 - \phi'_2.$
$\beta_{0,2} = \delta_2.$	$\beta_{0,2} = \delta_2.$
$h_2 = h_0.$	$h_2 = h_0.$
$a'_2 - r_2 = \dfrac{r_2 \phi'_2}{\beta_{0,2}}.$	$a'_2 - r_2 = \dfrac{r_2 \sin \phi'_2}{\sin \beta_{0,2}}.$
. . . .	. . . .
$\phi_{2t} = \dfrac{(a'_{2t-2} - r_{2t} - d_{2t-1})\beta_{0,2t-2}}{r_{2t}}.$	$\sin \phi_{2t} =$ $= \dfrac{(a'_{2t-2} - r_{2t} - d_{2t-1}) \sin \beta_{0,2t-2}}{r_{2t}}.$
$\phi'_{2t} = \phi_{2t} \dfrac{n_{2t-1}}{n_{2t+1}}.$	$\sin \phi'_{2t} = \sin \phi_{2t} \dfrac{n_{2t-1}}{n_{2t+1}}.$

(*b*) The first Radius infinite, the others not very large—*Cont.*

Paraxial Ray.	Rim Ray.
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$
$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$	$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$
$h_{2t} = r_{2t}(\phi'_{2t} + \beta_{0,2t}).$	$h_{2t} = r_{2t} \sin(\phi'_{2t} + \beta_{0,2t}).$
$a'_{2t} - r_{2t} = \frac{r_{2t}\phi'_{2t}}{\beta_{0,2t}}.$	$a'_{2t} - r_{2t} = \frac{r_{2t} \sin \phi'_{2t}}{\sin \beta_{0,2t}}.$
$v_{2t} = a'_{2t}.$	
$f_{0,2t} = \frac{h_0}{\beta_{0,2t}}.$	$f_{0,2t} = \frac{h_0}{\tan \beta_{0,2t}}.$
$e_{0,2t} = a'_{2t} - f_{0,2t}.$	$e_{0,2t} = a'_{2t} - f_{0,2t}.$

FOR AN INFINITELY DISTANT AXIAL OBJECT POINT

(*c*) The first Radius very large, the others not.

Paraxial Ray.	Rim Ray.
$\phi_0 = \frac{h_0}{r_0} = \eta_0.$	$\sin \phi_0 = \frac{h_0}{r_0} = \sin \eta_0.$
$\phi_0' = \phi_0 \frac{n_{-1}}{n_{+1}}.$	$\sin \phi_0' = \sin \phi_0 \frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$	$\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$
$a_0' = \frac{h_0}{\beta_{0,0}}.$	$a_0' = \frac{h_0}{\tan \beta_{0,0}} + 2 r_0 \sin^2 \frac{\eta_0}{2}.$
$\phi_2 = \frac{(a_0' - r_2 - d_1)\beta_{0,0}}{r_2}.$	$\sin \phi_2 = \frac{(a_0' - r_2 - d_1) \sin \beta_{0,0}}{r_2}.$
$\phi_2' = \phi_2 \frac{n_1}{n_3}.$	$\sin \phi_2' = \sin \phi_2 \frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\beta_{0,2} = \delta_2 + \beta_{0,0}.$	$\beta_{0,2} = \delta_2 + \beta_{0,0}.$
$h_2 = r_2(\phi_2' + \beta_{0,2}).$	$h_2 = r_2 \sin(\phi_2' + \beta_{0,2}).$

(*c*) The first Radius very large, the others not—*Cont.*

Paraxial Ray.	Rim Ray.
$a_2' - r_2 = \frac{r_2\phi_2'}{\beta_{0,2}}.$	$a_2' - r_2 = \frac{r_2 \sin\phi_2'}{\sin\beta_{0,2}}.$
. . . .	. . . .
	$\sin\phi_{2t} =$
$\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\beta_{0,2t-2}}{r_{2t}}.$	$= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\beta_{0,2t-2}}{r_{2t}}.$
$\phi'_{2t} = \phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$	$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$
$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$	$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$
$h_{2t} = r_{2t}(\phi'_{2t} + \beta_{0,2t}).$	$h_{2t} = r_{2t}\sin(\phi'_{2t} + \beta_{0,2t}).$
$a'_{2t} - r_{2t} = \frac{r_{2t}\phi'_{2t}}{\beta_{0,2t}}.$	$a'_{2t} - r_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\beta_{0,2t}}.$
$v_{2t} = a'_{2t}.$	
$f_{0,2t} = \frac{h_0}{\beta_{0,2t}}.$	$f_{0,2t} = \frac{h_0}{\tan\beta_{0,2t}}.$
$e_{0,2t} = a'_{2t} - f_{0,2t}.$	$e_{0,2t} = a'_{2t} - f_{0,2t}.$

FOR AN INFINITELY DISTANT AXIAL OBJECT POINT

(*d*) The $(s+1)$th Radius infinite, the others not very large.

Paraxial Ray.	Rim Ray.
$\phi_0 = \frac{h_0}{r_0}.$	$\sin\phi_0 = \frac{h_0}{r_0}.$
$\phi_0' = \phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'$	$\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$
$a_0' - r_0 = \frac{r_0\phi_0'}{\beta_{0,0}}.$	$a_0' - r_0 = \frac{r_0\sin\phi_0'}{\sin\beta_{0,0}}.$

(*d*) The $(s+1)^{\text{th}}$ Radius infinite, the others not very large—*Cont.*

Paraxial Ray.	Rim Ray.
$\phi_2 = \frac{(a_0' - r_2 - d_1)\beta_{0,0}}{r_2}.$	$\sin\phi_2 = \frac{(a_0' - r_2 - d_1)\sin\beta_{0,0}}{r_2}.$
$\phi_2' = \phi_2 \frac{n_1}{n_3}.$	$\sin\phi_2' = \sin\phi_2 \frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\beta_{0,2} = \delta_2 + \beta_{0,0}.$	$\beta_{0,2} = \delta_2 + \beta_{0,0}.$
$h_2 = r_2(\phi_2' + \beta_{0,2}).$	$h_2 = r_2 \sin(\phi_2' + \beta_{0,2}).$
$a_2' - r_2 = \frac{r_2\phi_2'}{\beta_{0,2}}.$	$a_2' - r_2 = \frac{r_2 \sin\phi_2'}{\sin\beta_{0,2}}.$
. . . .	. . . .
$\phi_{2s} = -\beta_{0,2s-2}.$	$\phi_{2s} = -\beta_{0,2s-2}.$
$\phi_{2s}' = \phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$	$\sin\phi_{2s}' = \sin\phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$
$\beta_{0,2s} = -\phi_{2s}'.$	$\beta_{0,2s} = -\phi_{2s}'.$
$a_{2s}' = \frac{(a_{2s-2}' - d_{2s-1})\beta_{0,2s-2}}{\beta_{0,2s}}.$	$a_{2s}' = \frac{(a_{2s-2}' - d_{2s-1})\tan\beta_{0,2s-2}}{\tan\beta_{0,2s}}.$
$h_{2s} = a_{2s}'\beta_{0,2s}.$	$h_{2s} = a_{2s}'\tan\beta_{0,2s}.$
. . . .	. . . .
$\phi_{2t} = \frac{(a_{2t-2}' - r_{2t} - d_{2t-1})\beta_{0,2t-2}}{r_{2t}}.$	$\sin\phi_{2t} =$ $= \frac{(a_{2t-2}' - r_{2t} - d_{2t-1})\sin\beta_{0,2t-2}}{r_{2t}}$
$\phi_{2t}' = \phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$	$\sin\phi_{2t}' = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi_{2t}'.$	$\delta_{2t} = \phi_{2t} - \phi_{2t}'.$
$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$	$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$
$h_{2t} = r_{2t}(\phi_{2t}' + \beta_{0,2t}).$	$h_{2t} = r_{2t}\sin(\phi_{2t}' + \beta_{0,2t}).$
$a_{2t}' - r_{2t} = \frac{r_{2t}\phi_{2t}'}{\beta_{0,2t}}.$	$a_{2t}' - r_{2t} = \frac{r_{2t}\sin\phi_{2t}'}{\sin\beta_{0,2t}}.$
$v_{2t} = a_{2t}'.$	
$f_{0,2t} = \frac{h_0}{\beta_{0,2t}}.$	$f_{0,2t} = \frac{h_0}{\tan\beta_{0,2t}}.$
$e_{0,2t} = a_{2t}' - f_{0,2t}.$	$e_{0,2t} = a_{2t}' - f_{0,2t}.$

FOR AN INFINITELY DISTANT AXIAL OBJECT POINT

(*e*) The $(s+1)^{\text{th}}$ Radius very large.

Paraxial Ray.	Rim Ray.
$\phi_0 = \frac{h_0}{r_0}.$	$\sin\phi_0 = \frac{h_0}{r_0}.$
$\phi_0' = \phi_0 \frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0 \frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$	$\delta_0 = \beta_{0,0} = \phi_0 - \phi_0'.$
$a_0' - r_0 = \frac{r_0\phi_0'}{\beta_{0,0}}.$	$a_0' - r_0 = \frac{r_0 \sin\phi_0'}{\sin\beta_{0,0}}.$
$\phi_2 = \frac{(a_0' - r_2 - d_1)\,\beta_{0,0}}{r_2}.$	$\sin\phi_2 = \frac{(a_0' - r_2 - d_1)\sin\beta_{0,0}}{r_2}.$
$\phi_2' = \phi_2 \frac{n_1}{n_3}.$	$\sin\phi_2' = \sin\phi_2 \frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\beta_{0,2} = \delta_2 + \beta_{0,0}.$	$\beta_{0,2} = \delta_2 + \beta_{0,0}.$
$h_2 = r_2(\phi_2' + \beta_{0,2}).$	$h_2 = r_2 \sin(\phi_2' + \beta_{0,2}).$
$a_2' - r_2 = \frac{r_2\phi_2'}{\beta_{0,2}}.$	$a_2' - r_2 = \frac{r_2 \sin\phi_2'}{\sin\beta_{0,2}}.$
. . . .	. . . .
$\phi_{2s} = \frac{(a'_{2s-2} - r_{2s} - d_{2s-1})\,\beta_{0,2s-2}}{r_{2s}}.$	$\sin\phi_{2s} =$ $= \frac{(a'_{2s-2} - r_{2s} - d_{2s-1})\sin\beta_{0,2s-2}}{r_{2s}}.$
$\phi'_{2s} = \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$	$\sin\phi'_{2s} = \sin\phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$
$\delta_{2s} = \phi_{2s} - \phi'_{2s}.$	$\delta_{2s} = \phi_{2s} - \phi'_{2s}.$
$\beta_{0,2s} = \delta_{2s} + \beta_{0,2s-2}.$	$\beta_{0,2s} = \delta_{2s} + \beta_{0,2s-2}.$
$\eta_{2s} = \phi_{2s} + \beta_{0,2s-2}.$	$\eta_{2s} = \phi_{2s} + \beta_{0,2s-2}.$
$h_{2s} = h_{2s-2} - d_{2s-1}\,\beta_{0,2s-2}.$	$h_{2s} = h_{2s-2} - (d_{2s-1} \mp$ $\mp \Delta d_{2s-2} \pm \Delta d_{2s}) \tan\beta_{0,2s-2}.$ †
	$\Delta d_{2s} = 2r_{2s} \sin^2 \frac{\eta_{2s}}{2}.$

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† Where the double sign occurs, the upper sign should be used when the radius having the corresponding suffix is positive, and vice versa.—ED.

(*e*) The $(s+1)^{\text{th}}$ Radius very large—*Cont.*

Paraxial Ray.	Rim Ray.
$a'_{2s} = \frac{h_{2s}}{\beta_{0,2s}}.$	$a'_{2s} = \frac{h_{2s}}{\tan\beta_{0,2s}} + 2r_{2s}\sin^2\frac{\eta_{2s}}{2}.$
. . . .	. . . .
$\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\,\beta_{0,2t-2}}{r_{2t}}.$	$\sin\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\beta_{0,2t-2}}{r_{2t}}.$
$\phi'_{2t} = \phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$	$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi'_{2t} - \phi_{2t}.$
$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$	$\beta_{0,2t} = \delta_{2t} + \beta_{0,2t-2}.$
$h_{2t} = r_{2t}(\phi'_{2t} + \beta_{0,2t}).$	$h_{2t} = r_{2t}\sin(\phi'_{2t} + \beta_{0,2t}).$
$a'_{2t} - r_{2t} = \frac{r_{2t}\phi'_{2t}}{\beta_{0,2t}}.$	$a'_{2t} - r_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\beta_{0,2t}}.$
$v_{2t} = a'_{2t}.$	
$f_{0,2t} = \frac{h_0}{\beta_{0,2t}}.$	$f_{0,2t} = \frac{h_0}{\tan\beta_{0,2t}}.$
$e_{0,2t} = a'_{2t} - f_{0,2t}.$	$e_{0,2t} = a'_{2t} - f_{0,2t}.$

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II. FOR RAYS PROCEEDING FROM AN INFINITELY DISTANT OBJECT POINT OUT OF THE AXIS

(*a*) For oblique Rays which lie in the axial plane.

(1) No Radius infinite or very large.	(2) The first Radius infinite.
$\left.\begin{aligned} \sin\phi_0 &= \frac{(e_{2t,0} - r_0)\sin\tau_0}{r_0}. \\ \eta_0 &= \phi_0 + \tau_0. \\ h_0 &= r_0 \sin\eta_0. \end{aligned}\right\}^{*}$	$h_0 = e_{2t,0}\tan\tau_0.^{*}$
$\sin\eta_0 = \frac{h_0}{r_0}.$	
$\phi_0 = \eta_0 - \alpha_0.$	$\phi_0 = -\alpha_0.$
$\sin\phi_0' = \sin\phi_0 \frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0 \frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \phi_0 - \phi_0'.$	$\delta_0 = \phi_0 - \phi_0'.$
$\alpha_0' = \delta_0 + \alpha_0.$	$\alpha_0' = \delta_0 + \alpha_0.$
$a_0' - r_0 = \frac{r_0 \sin\phi_0'}{\sin\alpha_0'}.$	$a_0' = \frac{h_0}{\tan\alpha_0'}.$
. . . .	. . . .
$\sin\phi_{2t} = \frac{(a_2{}'_{t-2} - r_{2t} - d_{2t-1})\sin\alpha_2{}'_{t-2}}{r_{2t}}.$	$\sin\phi_{2t} = \frac{(a_2{}'_{t-2} - r_{2t} - d_{2t-1})\sin\alpha_2{}'_{t-2}}{r_{2t}}.$
$\sin\phi_2{}'_t = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$	$\sin\phi_2{}'_t = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi_2{}'_t.$	$\delta_{2t} = \phi_{2t} - \phi_2{}'_t.$
$\alpha_2{}'_t = \delta_{2t} + \alpha_2{}'_{t-2}.$	$\alpha_2{}'_t = \delta_{2t} + \alpha_2{}'_{t-2}.$
$h_{2t} = r_{2t}\sin(\phi_2{}'_t + \alpha_2{}'_t).$	$h_{2t} = r_{2t}\sin(\phi_2{}'_t + \alpha_2{}'_t).$
$a_2{}'_t - r_{2t} = \frac{r_{2t}\sin\phi_2{}'_t}{\sin\alpha_2{}'_t}.$	$a_2{}'_t - r_{2t} = \frac{r_{2t}\sin\phi_2{}'_t}{\sin\alpha_2{}'_t}.$

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* Formulæ for the calculation of the approximate value of h_0 for the principal ray for which $\alpha_0 = \tau_0$, $\alpha_0' = \tau_0' \ldots \alpha_{2t} = \tau_{2t}$, $\alpha_2{}'_t = \tau_2{}'_t$.

(*a*) For oblique Rays which lie in the axial plane—*Cont.*

(1) No Radius infinite or very large.	(2) The first Radius infinite.
$h'_{2t} = (a'_{2t} - v'_{2t}) \tan \alpha'_{2t}.$	$h'_{2t} = (a'_{2t} - v'_{2t}) \tan \alpha'_{2t}.$
$\left.\begin{array}{l} x = \dfrac{\sin \phi_0 \, r_0 \tan \alpha_0}{\sin \alpha_0}. \\ z^2 = x^2 + y^2. \\ \cos \zeta_0 = \dfrac{x}{z}. \\ q_0 = z. \end{array}\right\} \dagger$	$\left.\begin{array}{l} x = h_0.^{*} \\ z^2 = x^2 + y^2. \\ \cos \zeta_0 = \dfrac{x}{z}. \\ q_0 = z. \end{array}\right\} \dagger$

(3) The first Radius very large.	(4) The $(s+1)$th Radius infinite.
$\left.\begin{array}{l} \sin \phi_0 = \dfrac{(e_{2t,0} - r_0) \sin \tau_0}{r_0}. \\ \eta_0 = \phi_0 + \tau_0. \\ h_0 = r_0 \sin \eta_0. \end{array}\right\} \ddagger$	$\left.\begin{array}{l} \sin \phi_0 = \dfrac{(e_{2t,0} - r_0) \sin \tau_0}{r_0}. \\ \eta_0 = \phi_0 + \tau_0. \\ h_0 = r_0 \sin \eta_0. \end{array}\right\} \ddagger$
$\sin \eta_0 = \dfrac{h_0}{r_0}.$	$\sin \eta_0 = \dfrac{h_0}{r_0}.$
$\phi_0 = \eta_0 - \alpha_0.$	$\phi_0 = \eta_0 - \alpha_0.$
$\sin \phi_0' = \sin \phi_0 \dfrac{n_{-1}}{n_{+1}}.$	$\sin \phi_0' = \sin \phi_0 \dfrac{n_{-1}}{n_{+1}}.$
$\delta_0 = \phi_0 - \phi_0'.$	$\delta_0 = \phi_0 - \phi_0'.$
$\alpha_0' = \delta_0 + \alpha_0.$	$\alpha_0' = \delta_0 + \alpha_0.$
$a_0' = \dfrac{h_0}{\tan \alpha_0'} + 2\, r_0 \sin^2 \dfrac{\eta_0}{2}.$	$a_0' - r_0 = \dfrac{r_0 \sin \phi_0'}{\sin \alpha_0'}.$
. . . .	. . . .
$\sin \phi_{2s} =$ $= \dfrac{(a'_{2s-2} - r_{2s} - d_{2s-1}) \sin \alpha'_{2s-2}}{r_{2s}}.$	$\phi_{2s} = -\alpha'_{2s-2}.$

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* In this case the perpendicular plane of reference is the surface itself.—ED.

† Formulæ for the determination of the constants for the rays not in the axial plane.

‡ Formulæ for the calculation of the approximate value of h_0 for the principal ray for which $\alpha_0 = \tau_0$, $\alpha_0' = \tau_0'$ . . . . $\alpha_{2t} = \tau_{2t}$, $\alpha'_{2t} = \tau'_{2t}$.

(*a*) For oblique Rays which lie in the axial plane—*Cont.*

(3) The first Radius very large.

$$\sin \phi'_{2s} = \sin \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$$

$$\delta_{2s} = \phi_{2s} - \phi'_{2s}.$$

$$\alpha'_{2s} = \delta_{2s} + \alpha'_{2s-2}.$$

$$h_{2s} = r_{2s} \sin (\phi'_{2s} + \alpha'_{2s}).$$

$$a'_{2s} - r_{2s} = \frac{r_{2s} \sin \phi'_{2s}}{\sin \alpha'_{2s}}.$$

. . . .

$$\sin \phi_{2t} =$$

$$= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1}) \sin \alpha'_{2t-2}}{r_{2t}}.$$

$$\sin \phi'_{2t} = \sin \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$$

$$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$$

$$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$$

$$h_{2t} = r_{2t} \sin (\phi'_{2t} + \alpha'_{2t}).$$

$$a'_{2t} - r_{2t} = \frac{r_{2t} \sin \phi'_{2t}}{\sin \alpha'_{2t}}.$$

$$h'_{2t} = (a'_{2t} - v'_{2t}) \tan \alpha'_{2t}.$$

$$\left.\begin{aligned} x &= \frac{\sin \phi_0 \, r_0 \tan \alpha_0}{\sin \alpha_0}. \\ z^2 &= x^2 + y^2. \\ \cos \zeta_0 &= \frac{x}{z}. \\ q_0 &= z. \end{aligned}\right\}^{*}$$

(4) The (*s* + 1)th Radius infinite.

$$\sin \phi'_{2s} = \sin \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$$

$$\alpha'_{2s} = -\phi'_{2s}.$$

$$a'_{2s} = \frac{(a'_{2s-2} - d_{2s-1}) \tan \alpha'_{2s-2}}{\tan \alpha'_{2s}}.$$

$$h_{2s} = a'_{2s} \tan \alpha'_{2s}.$$

. . . .

$$\sin \phi_{2t} =$$

$$= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1}) \sin \alpha'_{2t-2}}{r_{2t}}.$$

$$\sin \phi'_{2t} = \sin \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$$

$$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$$

$$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$$

$$h_{2t} = r_{2t} \sin (\phi'_{2t} + \alpha'_{2t}).$$

$$a'_{2t} - r_{2t} = \frac{r_{2t} \sin \phi'_{2t}}{\sin \alpha'_{2t}}.$$

$$h'_{2t} = (a'_{2t} - v'_{2t}) \tan \alpha'_{2t}.$$

$$\left.\begin{aligned} x &= \frac{\sin \phi_0 \, r_0 \tan \alpha_0}{\sin \alpha_0}. \\ z^2 &= x^2 + y^2. \\ \cos \zeta_0 &= \frac{x}{z}. \\ q_0 &= z. \end{aligned}\right\}^{*}$$

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* Formulæ for the determination of the constants for the rays not in the axial plane.

(*a*) For oblique Rays which lie in the axial plane—*Cont.*

(5) The (*s* + 1)th Radius very large.

$$\left.\begin{aligned}\sin\phi_0 &= \frac{(e_{2t,0}-r_0)\sin\tau_0}{\tau_0}.\\ \eta_0 &= \phi_0+\tau_0.\\ h_0 &= r_0\sin\eta_0.\end{aligned}\right\}^{*}$$

$$\sin\eta_0 = \frac{h_0}{r_0}.$$

$$\phi_0 = \eta_0-\alpha_0.$$

$$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$$

$$\delta_0 = \phi_0-\phi_0'.$$

$$\alpha_0' = \delta_0+\alpha_0.$$

$$a_0'-r_0 = \frac{r_0\sin\phi_0'}{\sin\alpha_0'}.$$

. . . .

$$\sin\phi_{2s} =$$

$$= \frac{(a'_{2s-2}-r_{2s}-d_{2s-1})\sin\alpha'_{2s-2}}{r_{2s}}.$$

$$\sin\phi'_{2s} = \sin\phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$$

$$\delta_{2s} = \phi_{2s}-\phi'_{2s}.$$

$$\alpha'_{2s} = \delta_{2s}+\alpha'_{2s-2}.$$

(5) The (*s* + 1)th Radius very large.

$$\eta_{2s} = \phi'_{2s}+\alpha'_{2s}.$$

$$h_{2s} = h_{2s-2}-$$

$$-(d_{2s-1}\mp\Delta d_{2s-2}\pm\Delta d_{2s})\tan\alpha'_{2s-2}.^{\dagger}$$

$$a'_{2s} = \frac{h_{2s}}{\tan\alpha'_{2s}}+2\,r_{2s}\sin^2\frac{\eta_{2s}}{2}.$$

. . . .

$$\sin\phi_{2t} = \frac{(a'_{2t-2}-r_{2t}-d_{2t-1})\sin\alpha'_{2t-2}}{r_{2t}}.$$

$$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$$

$$\delta_{2t} = \phi_{2t}-\phi'_{2t}.$$

$$\alpha'_{2t} = \delta_{2t}+\alpha'_{2t-2}.$$

$$h_{2t} = r_{2t}\sin(\phi'_{2t}+\alpha'_{2t}).$$

$$a'_{2t}-r_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\alpha'_{2t}}.$$

$$h'_{2t} = (a'_{2t}-v'_{2t})\tan\alpha'_{2t}.$$

$$\left.\begin{aligned}x &= \frac{\sin\phi_0\,r_0\tan\alpha_0}{\sin\alpha_0}.\\ z^2 &= x^2+y^2.\\ \cos\zeta_0 &= \frac{x}{z}.\\ q_0 &= z.\end{aligned}\right\}^{\ddagger}$$

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* Formulæ for the calculation of the approximate value of h_0 for the principal ray for which $\alpha_0=\tau_0$, $\alpha_0'=\tau_0' \ldots \alpha_{2t}=\tau_{2t}$, $\alpha'_{2t}=\tau'_{2t}$.

† Where the double sign occurs the upper sign should be used when the radius having the corresponding suffix is positive, and vice versa.—Ed.

‡ Formulæ for the determination of the constants for the rays not in the axial plane.

(*b*) For Rays which do not lie in the Axial Plane whether the Object is at a finite or infinite distance.

(1) No Radius Infinite.	(2) The First Radius Infinite.	(3) The $(s+1)^{\text{th}}$ Radius Infinite.
$\cos\lambda_0 = \mp\sin\tau_0\cos(\pi_0-\zeta_0).$*	$\cos\lambda_0 = \mp\sin\tau_0\cos(\pi_0-\zeta_0).$*	$\cos\lambda_0 = \mp\sin\tau_0\cos(\pi_0-\zeta_0).$*
$\sin\phi_0 = \frac{q_0\sin\lambda_0}{r_0}.$†	$\phi_0 = \tau_0.$	$\sin\phi_0 = \frac{q_0\sin\lambda_0}{r_0}.$†
$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$
$\lambda_0' = \lambda_0+\phi_0-\phi_0'.$	$\lambda_0' = \lambda_0+\phi_0-\phi_0'.$	$\lambda_0' = \lambda_0+\phi_0-\phi_0'.$
$q_0' = \frac{r_0\sin\phi_0'}{\sin\lambda_0'}\dagger = q_0\frac{n_{-1}}{n_{+1}}\frac{\sin\lambda_0}{\sin\lambda_0'}.$	$q_0' = q_0.$	$q_0' = \frac{r_0\sin\phi_0'}{\sin\lambda_0'}\dagger = q_0\frac{n_{-1}}{n_{+1}}\frac{\sin\lambda_0}{\sin\lambda_0'}.$
$\sin\tau_0'\sin(\pi_0'-\zeta_0) =$ $= \frac{\sin\lambda_0'}{\sin\lambda_0}\sin\tau_0\sin(\pi_0-\zeta_0).$	$\tau_0' = \phi_0'.$	$\sin\tau_0'\sin(\pi_0'-\zeta_0) =$ $= \frac{\sin\lambda_0'}{\sin\lambda_0}\sin\tau_0\sin(\pi_0-\zeta_0).$
$\sin\tau_0'\cos(\pi_0'-\zeta_0) = \mp\cos\lambda_0'.$*	$\pi_0' = \pi_0.$	$\sin\tau_0'\cos(\pi_0'-\zeta_0) = \mp\cos\lambda_0'.$*
‡ $\tan\tau_0' = \tan\tau_0\frac{\sin(\pi_0-\zeta_0)}{\sin(\pi_0'-\zeta_0)}$ $\pm\frac{\sin(\phi_0-\phi_0')}{\sin(\pi_0-\pi_0')} = \frac{\sin\lambda_0\sin\tau_0'}{\sin(\pi_0-\zeta_0)} =$ $= \frac{\sin\lambda_0'\sin\tau_0}{\sin(\pi_0'-\zeta_0)}.$*	‡ $\sin\lambda_0\sin\tau_0' = \sin\lambda_0'\sin\tau_0.$	‡ $\tan\tau_0' = \tan\tau_0\frac{\sin(\pi_0-\zeta_0)}{\sin(\pi_0'-\zeta_0)}$ $\pm\frac{\sin(\phi_0-\phi_0')}{\sin(\pi_0-\pi_0')} = \frac{\sin\lambda_0\sin\tau_0'}{\sin(\pi_0-\zeta_0)} =$ $= \frac{\sin\lambda_0'\sin\tau_0}{\sin(\pi_0'-\zeta_0)}.$*
$\sin\epsilon_0 = \sin\tau_0'\sin\pi_0'.$	$\sin\epsilon_0 = \sin\tau_0'\sin\pi_0'.$	$\sin\epsilon_0 = \sin\tau_0'\sin\pi_0'.$
$\frac{\sin(\phi_0-\phi_0')\sin(\phi_0+\phi_0')}{\sin\phi_0\sin\phi_0'} = 2\cot 2\omega.$	$\frac{\sin(\phi_0-\phi_0')\sin(\phi_0+\phi_0')}{\sin\phi_0\sin\phi_0'} = 2\cot 2\omega.$	$\frac{\sin(\phi_0-\phi_0')\sin(\phi_0+\phi_0')}{\sin\phi_0\sin\phi_0'} = 2\cot 2\omega.$

$\tau_2 = \tau_0'.$	$\tau_2 = \tau_0.$	$\tau_2 = \tau_0'.$
$\pi_2 = \pi_0'.$	$\pi_2 = \pi_0'$	$\pi_2 = \pi_0'.$
$c_1 = d_1 - r_0 + r_2.$	$c_1 = d_1 + r_2.$	$c_1 = d_1 - r_0 + r_2.$
$q_2 \sin(\pi_0' - \zeta_2) = q_0' \sin(\pi_0' - \zeta_0).$	$q_2 \sin(\pi_0' - \zeta_2) = q_0' \sin(\pi_0' - \zeta_0).$	$q_2 \sin(\pi_0' - \zeta_2) = q_0' \sin(\pi_0' - \zeta_0).$
$q_2 \cos(\pi_0' - \zeta_2) = q_0' \cos(\pi_0' - \zeta_0) -$ $- c_1 \tan \tau_0'.$	$q_2 \cos(\pi_0' - \zeta_2) = q_0' \cos(\pi_0' - \zeta_0) -$ $- c_1 \tan \tau_0'.$	$q_2 \cos(\pi_0' - \zeta_2) = q_0' \cos(\pi_0' - \zeta_0) -$ $- c_1 \tan \tau_0'.$
$\frac{c_1 \tan \tau_0'}{\sin(\zeta_0 - \zeta_2)} = \frac{q_2}{\sin(\pi_0' - \zeta_0)} = \frac{q_0'}{\sin(\pi_0' - \zeta_2)}.^{\ddagger}$	$\frac{c_1 \tan \tau_0'}{\sin(\zeta_0 - \zeta_2)} = \frac{q_2}{\sin(\pi_0' - \zeta_0)} = \frac{q_0'}{\sin(\pi_0' - \zeta_2)}.^{\ddagger}$	$\frac{c_1 \tan \tau_0'}{\sin(\zeta_0 - \zeta_2)} = \frac{q_2}{\sin(\pi_0' - \zeta_0)} = \frac{q_0'}{\sin(\pi_0' - \zeta_2)}.^{\ddagger}$
. . . .	. . . .	. . . .
$\cos \lambda_{2s} = \mp \sin \tau_{2s} \cos(\pi_{2s} - \zeta_{2s}).^{*}$	$\cos \lambda_{2s} = \mp \sin \tau_{2s} \cos(\pi_{2s} - \zeta_{2s}).^{*}$	$\cos \lambda_{2s} = \mp \sin \tau_{2s} \cos(\pi_{2s} - \zeta_{2s}).^{*}$
$\sin \phi_{2s} = \frac{q_{2s} \sin \lambda_{2s}{}^{\dagger}}{r_{2s}}.$	$\sin \phi_{2s} = \frac{q_{2s} \sin \lambda_{2s}{}^{\dagger}}{r_{2s}}.$	$\phi_{2s} = \tau_{2s}.$
$\sin \phi_{2s}' = \sin \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$	$\sin \phi_{2s}' = \sin \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$	$\sin \phi_{2s}' = \sin \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$
$\lambda_{2s}' = \lambda_{2s} + \phi_{2s} - \phi_{2s}'.$	$\lambda_{2s}' = \lambda_{2s} + \phi_{2s} - \phi_{2s}'.$	$\lambda_{2s}' = \lambda_{2s} + \phi_{2s} - \phi_{2s}'.$
$q_{2s}' = \frac{r_{2s} \sin \phi_{2s}'{}^{\dagger}}{\sin \lambda_{2s}'} = q_{2s} \frac{n_{2s-1}}{n_{2s+1}} \frac{\sin \lambda_{2s}}{\sin \lambda_{2s}'}.$	$q_{2s}' = \frac{r_{2s} \sin \phi_{2s}'{}^{\dagger}}{\sin \lambda_{2s}'} = q_{2s} \frac{n_{2s-1}}{n_{2s+1}} \frac{\sin \lambda_{2s}}{\sin \lambda_{2s}'}.$	$q_{2s}' = q_{2s}.$
$\sin \tau_{2s}' \sin(\pi_{2s}' - \zeta_{2s}) =$ $= \frac{\sin \lambda_{2s}'}{\sin \lambda_{2s}} \sin \tau_{2s} \sin(\pi_{2s} - \zeta_{2s}).$	$\sin \tau_{2s}' \sin(\pi_{2s}' - \zeta_{2s}) =$ $= \frac{\sin \lambda_{2s}'}{\sin \lambda_{2s}} \sin \tau_{2s} \sin(\pi_{2s} - \zeta_{2s}).$	$\tau_{2s}' = \phi_{2s}'.$

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* Where the double sign occurs the upper sign should be used when the radius having the corresponding suffix is positive, and vice versa.—ED.

† In these cases r is always positive whether the surface is convex or concave.—ED.

‡ These formulæ serve as checks.

(*b*) For Rays which do not lie in the Axial Plane whether the Object is at a finite or infinite distance—*Continued.*

(1) No Radius Infinite.	(2) The First Radius Infinite.	(3) The $(s+1)^{\text{th}}$ Radius Infinite.
$\sin\tau'_{2s}\cos(\pi'_{2s}-\zeta_{2s}) = \mp\cos\lambda'_{2s}.$*	$\sin\tau'_{2s}\cos(\pi'_{2s}-\zeta_{2s}) = \mp\cos\lambda'_{2s}.$*	$\pi'_{2s} = \pi_{2s}.$
$\left\{\begin{array}{l} \tan\tau'_{2s} = \tan\tau_{2s}\dfrac{\sin(\pi_{2s}-\zeta_{2s})}{\sin(\pi'_{2s}-\zeta_{2s})} \\ \quad\pm\dfrac{\sin(\phi_{2s}-\phi'_{2s})}{\sin(\pi_{2s}-\pi'_{2s})} = \\ \quad=\dfrac{\sin\lambda_{2s}\sin\tau'_{2s}}{\sin(\pi_{2s}-\zeta_{2s})} = \dfrac{\sin\lambda'_{2s}\sin\tau_{2s}}{\sin(\pi'_{2s}-\zeta_{2s})}.^{*} \\ \sin\epsilon_{2s} = \sin\tau'_{2s}\sin\pi'_{2s}. \\ \dfrac{\sin(\phi_{2s}-\phi'_{2s})\sin(\phi_{2s}+\phi'_{2s})}{\sin\phi_{2s}\sin\phi'_{2s}} = \\ \quad= 2\cot 2\omega. \end{array}\right\}$ ‡	$\left\{\begin{array}{l} \tan\tau'_{2s} = \tan\tau_{2s}\dfrac{\sin(\pi_{2s}-\zeta_{2s})}{\sin(\pi'_{2s}-\zeta_{2s})} \\ \quad\pm\dfrac{\sin(\phi_{2s}-\phi'_{2s})}{\sin(\pi_{2s}-\pi'_{2s})} = \\ \quad=\dfrac{\sin\lambda_{2s}\sin\tau'_{2s}}{\sin(\pi_{2s}-\zeta_{2s})} = \dfrac{\sin\lambda'_{2s}\sin\tau_{2s}}{\sin(\pi'_{2s}-\zeta_{2s})}.^{*} \\ \sin\epsilon_{2s} = \sin\tau'_{2s}\sin\pi'_{2s}. \\ \dfrac{\sin(\phi_{2s}-\phi'_{2s})\sin(\phi_{2s}+\phi'_{2s})}{\sin\phi_{2s}\sin\phi'_{2s}} = \\ \quad= 2\cot 2\omega. \end{array}\right\}$ ‡	$\left\{\begin{array}{l} \sin\lambda_{2s}\sin\tau'_{2s} = \sin\lambda'_{2s}\sin\tau_{2s}. \\ \sin\epsilon_{2s} = \sin\tau'_{2s}\sin\pi'_{2s}. \\ \dfrac{\sin(\phi_{2s}-\phi'_{2s})\sin(\phi_{2s}+\phi'_{2s})}{\sin\phi_{2s}\sin\phi'_{2s}} = \\ \quad= 2\cot 2\omega. \end{array}\right\}$ ‡
$\tau_{2s+2} = \tau'_{2s}.$	$\tau_{2s+2} = \tau'_{2s}.$	$\tau_{2s+2} = \tau'_{2s}.$
$\pi_{2s+2} = \pi'_{2s}.$	$\pi_{2s+2} = \pi'_{2s}.$	$\pi_{2s+2} = \pi'_{2s}.$
$c_{2s+1} = d_{2s+1} - r_{2s} + r_{2s+2}.$	$c_{2s+1} = d_{2s+1} - r_{2s} + r_{2s+2}.$	$c_{2s+1} = d_{2s+1} + r_{2s+2}.$
$q_{2s+2}\sin(\pi'_{2s}-\zeta_{2s+2}) = q'_{2s}\sin(\pi'_{2s}-\zeta_{2s}).$	$q_{2s+2}\sin(\pi'_{2s}-\zeta_{2s+2}) = q'_{2s}\sin(\pi'_{2s}-\zeta_{2s}).$	$q_{2s+2}\sin(\pi'_{2s}-\zeta_{2s+2}) = q'_{2s}\sin(\pi'_{2s}-\zeta_{2s}).$
$q_{2s+2}\cos(\pi'_{2s}-\zeta_{2s+2}) =$ $= q'_{2s}\cos(\pi'_{2s}-\zeta_{2s}) - c_{2s+1}\tan\tau'_{2s}.$	$q_{2s+2}\cos(\pi'_{2s}-\zeta_{2s+2}) =$ $= q'_{2s}\cos(\pi'_{2s}-\zeta_{2s}) - c_{2s+1}\tan\tau'_{2s}.$	$q_{2s+2}\cos(\pi'_{2s}-\zeta_{2s+2}) =$ $= q'_{2s}\cos(\pi'_{2s}-\zeta_{2s}) - c_{2s+1}\tan\tau'_{2s}.$

$$\left\{\begin{aligned}\frac{c_{2s+1}\tan\tau'_{2s}}{\sin(\zeta_{2s}-\zeta_{2s+2})} &= \frac{q_{2s+2}}{\sin(\pi'_{2s}-\zeta_{2s})} = \\ &= \frac{q'_{2s}}{\sin(\pi'_{2s}-\zeta_{2s+2})}.\end{aligned}\right\}\ddagger$$

. . . .

$$\cos\lambda_{2t} = \mp\sin\tau_{2t}\cos(\pi_{2t}-\zeta_{2t}).^{*}$$

$$\sin\phi_{2t} = \frac{q_{2t}\sin\lambda_{2t}}{r_{2t}}.^{\dagger}$$

$$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$$

$$\lambda'_{2t} = \lambda_{2t}+\phi_{2t}-\phi'_{2t}.$$

$$q'_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\lambda'_{2t}}^{\dagger} = q_{2t}\frac{n_{2t-1}}{n_{2t+1}}\frac{\sin\lambda_{2t}}{\sin\lambda'_{2t}}.$$

$$\sin\tau'_{2t}\sin(\pi'_{2t}-\zeta_{2t}) = $$

$$= \frac{\sin\lambda'_{2t}}{\sin\lambda_{2t}}\sin\tau_{2t}\sin(\pi_{2t}-\zeta_{2t}).$$

$$\sin\tau'_{2t}\cos(\pi'_{2t}-\zeta_{2t}) = \mp\cos\lambda'_{2t}.^{*}$$

$$\left\{\begin{aligned}\frac{c_{2s+1}\tan\tau'_{2s}}{\sin(\zeta_{2s}-\zeta_{2s+2})} &= \frac{q_{2s+2}}{\sin(\pi'_{2s}-\zeta_{2s})} = \\ &= \frac{q'_{2s}}{\sin(\pi'_{2s}-\zeta_{2s+2})}.\end{aligned}\right\}\ddagger$$

. . . .

$$\cos\lambda_{2t} = \mp\sin\tau_{2t}\cos(\pi_{2t}-\zeta_{2t}).^{*}$$

$$\sin\phi_{2t} = \frac{q_{2t}\sin\lambda_{2t}}{r_{2t}}.^{\dagger}$$

$$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$$

$$\lambda'_{2t} = \lambda_{2t}+\phi_{2t}-\phi'_{2t}.$$

$$q'_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\lambda'_{2t}}^{\dagger} = q_{2t}\frac{n_{2t-1}}{n_{2t+1}}\frac{\sin\lambda_{2t}}{\sin\lambda'_{2t}}.$$

$$\sin\tau'_{2t}\sin(\pi'_{2t}-\zeta_{2t}) = $$

$$= \frac{\sin\lambda'_{2t}}{\sin\lambda_{2t}}\sin\tau_{2t}\sin(\pi_{2t}-\zeta_{2t}).$$

$$\sin\tau'_{2t}\cos(\pi'_{2t}-\zeta_{2t}) = \mp\cos\lambda'_{2t}.^{*}$$

$$\left\{\begin{aligned}\frac{c_{2s+1}\tan\tau'_{2s}}{\sin(\zeta_{2s}-\zeta_{2s+2})} &= \frac{q_{2s+2}}{\sin(\pi'_{2s}-\zeta_{2s})} = \\ &= \frac{q'_{2s}}{\sin(\pi'_{2s}-\zeta_{2s+2})}.\end{aligned}\right\}\ddagger$$

. . . .

$$\cos\lambda_{2t} = \mp\sin\tau_{2t}\cos(\pi_{2t}-\zeta_{2t}).^{*}$$

$$\sin\phi_{2t} = \frac{q_{2t}\sin\lambda_{2t}}{r_{2t}}.^{\dagger}$$

$$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$$

$$\lambda'_{2t} = \lambda_{2t}+\phi_{2t}-\phi'_{2t}.$$

$$q'_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\lambda'_{2t}}^{\dagger} = q_{2t}\frac{n_{2t-1}}{n_{2t+1}}\frac{\sin\lambda_{2t}}{\sin\lambda'_{2t}}.$$

$$\sin\tau'_{2t}\sin(\pi'_{2t}-\zeta_{2t}) = $$

$$= \frac{\sin\lambda'_{2t}}{\sin\lambda_{2t}}\sin\tau_{2t}\sin(\pi_{2t}-\zeta_{2t}).$$

$$\sin\tau'_{2t}\cos(\pi'_{2t}-\zeta_{2t}) = \mp\cos\lambda'_{2t}.^{*}$$

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* Where the double sign occurs the upper sign should be used when the radius having the corresponding suffix is positive, and vice versa.—ED.

† In these cases *r* is always positive whether the surface is convex or concave.—ED.

‡ These formulæ serve as checks.

(*b*) For Rays which do not lie in the Axial Plane whether the Object is at a finite or infinite distance.—*Continued.*

(1) No Radius Infinite.	(2) The First Radius Infinite.	(3) The $(s+1)^{\text{th}}$ Radius Infinite.
$\left\{\begin{array}{l} \tan\tau'_{2t} = \tan\tau_{2t}\dfrac{\sin(\pi_{2t}-\zeta_{2t})}{\sin(\pi'_{2t}-\zeta_{2t})} \\ \quad\pm\dfrac{\sin(\phi_{2t}-\phi'_{2t})}{\sin(\pi_{2t}-\pi'_{2t})} = \dfrac{\sin\lambda_{2t}\sin\tau'_{2t}}{\sin(\pi_{2t}-\zeta_{2t})} = \\ \quad = \dfrac{\sin\lambda'_{2t}\sin\tau_{2t}}{\sin(\pi'_{2t}-\zeta_{2t})}.^{*} \\ \sin\epsilon_{2t} = \sin\tau'_{2t}\sin\pi'_{2t}. \\ \dfrac{\sin(\phi_{2t}-\phi'_{2t})\sin(\phi_{2t}+\phi'_{2t})}{\sin\phi_{2t}\sin\phi'_{2t}} = \\ \quad = 2\cot 2\omega. \end{array}\right\}\ddagger$ $c_{2t+1} = b'_{2t} - r_{2t}.\S$ $z\sin(\pi'_{2t}-\xi) = q'_{2t}\sin(\pi'_{2t}-\zeta_{2t}).\S$ $z\cos(\pi'_{2t}-\xi) =$ $\quad = q'_{2t}\cos(\pi'_{2t}-\zeta_{2t}) - c_{2t+1}\tan\tau'_{2t}.\S$ $\left\{\begin{array}{l} \dfrac{c_{2t+1}\tan\tau'_{2t}}{\sin(\zeta_{2t}-\xi)} = \dfrac{z}{\sin(\pi'_{2t}-\zeta_{2t})} = \\ \quad = \dfrac{q'_{2t}}{\sin(\pi'_{2t}-\xi)}. \end{array}\right\}\ddagger$	$\left\{\begin{array}{l} \tan\tau'_{2t} = \tan\tau_{2t}\dfrac{\sin(\pi_{2t}-\zeta_{2t})}{\sin(\pi'_{2t}-\zeta_{2t})} \\ \quad\pm\dfrac{\sin(\phi_{2t}-\phi'_{2t})}{\sin(\pi_{2t}-\pi'_{2t})} = \dfrac{\sin\lambda_{2t}\sin\tau'_{2t}}{\sin(\pi_{2t}-\zeta_{2t})} = \\ \quad = \dfrac{\sin\lambda'_{2t}\sin\tau_{2t}}{\sin(\pi'_{2t}-\zeta_{2t})}.^{*} \\ \sin\epsilon_{2t} = \sin\tau'_{2t}\sin\pi'_{2t}. \\ \dfrac{\sin(\phi_{2t}-\phi'_{2t})\sin(\phi_{2t}+\phi'_{2t})}{\sin\phi_{2t}\sin\phi'_{2t}} = \\ \quad = 2\cot 2\omega. \end{array}\right\}\ddagger$ $c_{2t+1} = b'_{2t} - r_{2t}.\S$ $z\sin(\pi'_{2t}-\xi) = q'_{2t}\sin(\pi'_{2t}-\zeta_{2t}).\S$ $z\cos(\pi'_{2t}-\xi) =$ $\quad = q'_{2t}\cos(\pi'_{2t}-\zeta_{2t}) - c_{2t+1}\tan\tau'_{2t}.\S$ $\left\{\begin{array}{l} \dfrac{c_{2t+1}\tan\tau'_{2t}}{\sin(\zeta_{2t}-\xi)} = \dfrac{z}{\sin(\pi'_{2t}-\zeta_{2t})} = \\ \quad = \dfrac{q'_{2t}}{\sin(\pi'_{2t}-\xi)}.\S \end{array}\right\}\ddagger$	$\left\{\begin{array}{l} \tan\tau'_{2t} = \tan\tau_{2t}\dfrac{\sin(\pi_{2t}-\zeta_{2t})}{\sin(\pi'_{2t}-\zeta_{2t})} \\ \quad\pm\dfrac{\sin(\phi_{2t}-\phi'_{2t})}{\sin(\pi_{2t}-\pi'_{2t})} = \dfrac{\sin\lambda_{2t}\sin\tau'_{2t}}{\sin(\pi_{2t}-\zeta_{2t})} = \\ \quad = \dfrac{\sin\lambda'_{2t}\sin\tau_{2t}}{\sin(\pi'_{2t}-\zeta_{2t})}.^{*} \\ \sin\epsilon_{2t} = \sin\tau'_{2t}\sin\pi'_{2t}. \\ \dfrac{\sin(\phi_{2t}-\phi'_{2t})\sin(\phi_{2t}+\phi'_{2t})}{\sin\phi_{2t}\sin\phi'_{2t}} = \\ \quad = 2\cot 2\omega. \end{array}\right\}\ddagger$ $c_{2t+1} = b'_{2t} - r_{2t}.\S$ $z\sin(\pi'_{2t}-\xi) = q'_{2t}\sin(\pi'_{2t}-\zeta_{2t}).\S$ $z\cos(\pi'_{2t}-\xi) =$ $\quad = q'_{2t}\cos(\pi'_{2t}-\zeta_{2t}) - c_{2t+1}\tan\tau'_{2t}.\S$ $\left\{\begin{array}{l} \dfrac{c_{2t+1}\tan\tau'_{2t}}{\sin(\zeta_{2t}-\xi)} = \dfrac{z}{\sin(\pi'_{2t}-\zeta_{2t})} = \\ \quad = \dfrac{q'_{2t}}{\sin(\pi'_{2t}-\xi)}.\S \end{array}\right\}\ddagger$

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* Where the double sign occurs the upper sign should be used when the radius having the corresponding suffix is positive, and vice versa.—Ed.

‡ These formulæ serve as checks.

§ These formulæ refer to a final plane which is not necessarily a centre plane. The interval c then represents the separation of the final plane from the previous centre plane. The polar co-ordinates of the point of intersection of the emergent ray with the final plane are z and ξ. See fig. 8.—Ed.

III. FOR RAYS WHICH PROCEED FROM AN AXIAL OBJECT POINT AT A FINITE DISTANCE

(*a*) No Radius very large.

Paraxial Ray.	Rim Ray.
$\eta_0 = \frac{h_0}{r_0}.$	$\sin \eta_0 = \frac{h_0}{r_0}.$
$\alpha_0 = \frac{h_0}{u}.$	$\tan \alpha_0 = \frac{h_0}{u - 2r_0 \sin^2 \frac{\eta_0}{2}}.$
$\phi_0 = \eta_0 - \alpha_0.$	$\phi_0 = \eta_0 - \alpha_0.$
$\phi_0' = \phi_0 \frac{n_{-1}}{n_{+1}}.$	$\sin \phi_0' = \sin \phi_0 \frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \phi_0 - \phi_0'.$	$\delta_0 = \phi_0 - \phi_0'.$
$\alpha_0' = \delta_0 + \alpha_0.$	$\alpha_0' = \delta_0 + \alpha_0.$
$a_0' - r_0 = \frac{r_0 \phi_0'}{\alpha_0'}.$	$a_0' - r_0 = \frac{r_0 \sin \phi_0'}{\sin \alpha_0'}.$
$\phi_2 = \frac{(a_0' - r_2 - d_1)\alpha_0'}{r_2}.$	$\sin \phi_2 = \frac{(a_0' - r_2 - d_1) \sin \alpha_0'}{r_2}.$
$\phi_2' = \phi_2 \frac{n_1}{n_3}.$	$\sin \phi_2' = \sin \phi_2 \frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\alpha_2' = \delta_2 + \alpha_0'.$	$\alpha_2' = \delta_2 + \alpha_0'.$
$h_2 = r_2(\phi_2' + \alpha_2').$	$h_2 = r_2 \sin (\phi_2' + \alpha_2').$
$a_2' - r_2 = \frac{r_2 \phi_2'}{\alpha_2'}.$	$a_2' - r_2 = \frac{r_2 \sin \phi_2'}{\sin \alpha_2'}.$
. . . .	. . . .
$\phi_{2s} = \frac{(a_{2s-2}' - r_{2s} - d_{2s-1})\, \alpha'_{2s-2}}{r_{2s}}.$	$\sin \phi_{2s} =$ $= \frac{(a_{2s-2}' - r_{2s} - d_{2s-1}) \sin \alpha_{2s-2}'}{r_{2s}}.$
$\phi_{2s}' = \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$	$\sin \phi_{2s}' = \sin \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$
$\delta_{2s} = \phi_{2s} - \phi_{2s}'.$	$\delta_{2s} = \phi_{2s} - \phi_{2s}'.$

(*a*) No Radius very large—*Cont.*

Paraxial Ray.	Rim Ray.
$\alpha'_{2s} = \delta_{2s} + \alpha'_{2s-2}.$	$\alpha'_{2s} = \delta_{2s} + \alpha'_{2s-2}.$
$h_{2s} = r_{2s}(\phi'_{2s} + \alpha'_{2s}).$	$h_{2s} = r_{2s}\sin(\phi'_{2s} + \alpha'_{2s}).$
$a'_{2s} - r_{2s} = \frac{r_{2s}\phi'_{2s}}{\alpha'_{2s}}.$	$a'_{2s} - r_{2s} = \frac{r_{2s}\sin\phi'_{2s}}{\sin\alpha'_{2s}}.$
. . . .	. . . .
$\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\,\alpha'_{2t-2}}{r_{2t}}.$	$\sin\phi_{2t} =$ $= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\alpha'_{2t-2}}{r_{2t}}.$
$\phi'_{2t} = \phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$	$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$
$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$	$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$
$h_{2t} = r_{2t}(\phi'_{2t} + \alpha'_{2t}).$	$h_{2t} = r_{2t}\sin(\phi'_{2t} + \alpha'_{2t}).$
$a'_{2t} - r_{2t} = \frac{r_{2t}\phi'_{2t}}{\alpha'_{2t}}.$	$a'_{2t} - r_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\alpha'_{2t}}.$

(*b*) The first Radius infinite, the others not very large.

Paraxial Ray.	Rim Ray.
$\alpha_0 = \frac{h_0}{u}.$	$\tan\alpha_0 = \frac{h_0}{u}.$
$\phi_0 = -\alpha_0.$	$\phi_0 = -\alpha_0.$
$\phi'_0 = \phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi'_0 = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \phi_0 - \phi'_0.$	$\delta_0 = \phi_0 - \phi'_0.$
$\alpha'_0 = \delta_0 + \alpha_0.$	$\alpha'_0 = \delta_0 + \alpha_0.$
$a'_0 = u\frac{n_{+1}}{n_{-1}}.$	$a'_0 = u\frac{\tan\alpha_0}{\tan\alpha'_0}.$
$\phi_2 = \frac{(a'_0 - r_2 - d_1)\alpha'_0}{r_2}.$	$\sin\phi_2 = \frac{(a'_0 - r_2 - d_1)\sin\alpha'_0}{r_2}.$

(*b*) The first Radius infinite, the others not very large—*Cont.*

Paraxial Ray.	Rim Ray.
$\phi_2' = \phi_2 \frac{n_1}{n_3}.$	$\sin \phi_2' = \sin \phi_2 \frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\alpha_2' = \delta_2 + \alpha_0'.$	$\alpha_2' = \delta_2 + \alpha_0'.$
$h_2 = r_2(\phi_2' + \alpha_2').$	$h_2 = r_2 \sin(\phi_2' + \alpha_2').$
$a_2' - r_2 = \frac{r_2 \phi_2'}{\alpha_2'}.$	$a_2' - r_2 = \frac{r_2 \sin \phi_2'}{\sin \alpha_2'}.$
. . . .	. . . .
$\phi_{2s} = \frac{(a'_{2\,s-2} - r_{2s} - d_{2s-1})\alpha'_{2\,s-2}}{r_{2s}}.$	$\sin \phi_{2s} = \frac{(a'_{2\,s-2} - r_{2s} - d_{2s-1}) \sin \alpha'_{2\,s-2}}{r_{2s}}.$
$\phi'_{2s} = \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}.$	$\sin \phi'_{2s} = \sin \phi_{2s} \cdot \frac{n_{2s-1}}{n_{2s+1}}.$
$\delta_{2s} = \phi_{2s} - \phi'_{2s}.$	$\delta_{2s} = \phi_{2s} - \phi'_{2s}.$
$\alpha'_{2s} = \delta_{2s} + \alpha'_{2\,s-2}.$	$\alpha'_{2s} = \delta_{2s} + \alpha'_{2\,s-2}.$
$h_{2s} = r_{2s}(\phi'_{2s} + \alpha'_{2s}).$	$h_{2s} = r_{2s} \sin(\phi'_{2s} + \alpha'_{2s}).$
$a'_{2s} - r_{2s} = \frac{r_{2s}\phi'_{2s}}{\alpha'_{2s}}.$	$a'_{2s} - r_{2s} = \frac{r_{2s} \sin \phi'_{2s}}{\sin \alpha'_{2s}}.$
. . . .	. . . .
$\phi_{2t} = \frac{(a'_{2\,t-2} - r_{2t} - d_{2t-1})\alpha'_{2\,t-2}}{r_{2t}}.$	$\sin \phi_{2t} = \frac{(a'_{2\,t-2} - r_{2t} - d_{2t-1}) \sin \alpha'_{2\,t-2}}{r_{2t}}.$
$\phi'_{2t} = \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$	$\sin \phi'_{2t} = \sin \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$
$\alpha'_{2t} = \delta_{2t} + \alpha'_{2\,t-2}.$	$\alpha'_{2t} = \delta_{2t} + \alpha'_{2\,t-2}.$
$h_{2t} = r_{2t}(\phi'_{2t} + \alpha'_{2t}).$	$h_{2t} = r_{2t} \sin(\phi'_{2t} + \alpha'_{2t}).$
$a'_{2t} - r_{2t} = \frac{r_{2t}\phi'_{2t}}{\alpha'_{2t}}.$	$a'_{2t} - r_{2t} = \frac{r_{2t} \sin \phi'_{2t}}{\sin \alpha'_{2t}}.$

(*c*) The first Radius very large.

Paraxial Ray.	Rim Ray.
$\eta_0 = \frac{h_0}{r_0}.$	$\sin\eta_0 = \frac{h_0}{r_0}.$
$\alpha_0 = \frac{h_0}{u}.$	$\tan\alpha_0 = \frac{h_0}{u - 2r_0\sin^2\frac{\eta_0}{2}}.$
$\phi_0 = \eta_0 - \alpha_0.$	$\phi_0 = \eta_0 - \alpha_0.$
$\phi_0' = \phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \phi_0 - \phi_0'.$	$\delta_0 = \phi_0 - \phi_0'.$
$\alpha_0' = \delta_0 + \alpha_0.$	$\alpha_0' = \delta_0 + \alpha_0.$
$a_0' = \frac{h_0}{\alpha_0'}.$	$a_0' = \frac{h_0}{\tan\alpha_0'} - 2r_0\sin^2\frac{\eta_0}{2}.$
$\phi_2 = \frac{(a_0' - r_2 - d_1)\alpha_0'}{r_2}.$	$\sin\phi_2 = \frac{(a_0' - r_2 - d_1)\sin\alpha_0'}{r_2}.$
$\phi_2' = \phi_2\frac{n_1}{n_3}.$	$\sin\phi_2' = \sin\phi_2\frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\alpha_2' = \delta_2 + \alpha_0'.$	$\alpha_2' = \delta_2 + \alpha_0'.$
$h_2 = r_2(\phi_2' + \alpha_2').$	$h_2 = r_2\sin(\phi_2' + \alpha_2').$
$a_2' - r_2 = \frac{r_2\phi_2'}{\alpha_2'}.$	$a_2' - r_2 = \frac{r_2\sin\phi_2'}{\sin\alpha_2'}.$
. . . .	. . . . . .
$\phi_{2s} = \frac{(a_{2\,s-2}' - r_{2s} - d_{2s-1})\alpha_{2\,s-2}'}{r_{2s}}.$	$\sin\phi_{2s} = \frac{(a_{2\,s-2}' - r_{2s} - d_{2s-1})\sin\alpha_{2\,s-2}'}{r_{2s}}.$
$\phi_{2s}' = \phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$	$\sin\phi_{2s}' = \sin\phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$
$\delta_{2s} = \phi_{2s} - \phi_{2s}'.$	$\delta_{2s} = \phi_{2s} - \phi_{2s}'.$
$\alpha_{2s}' = \delta_{2s} + \alpha_{2\,s-2}'.$	$\alpha_{2s}' = \delta_{2s} + \alpha_{2\,s-2}'.$
$h_{2s} = r_{2s}(\phi_{2s}' + \alpha_{2s}').$	$h_{2s} = r_{2s}\sin(\phi_{2s}' + \alpha_{2s}').$

(*c*) The first Radius very large—*Cont.*

Paraxial Ray.	Rim Ray.
$a_{2s}' - r_{2s} = \frac{r_{2s}\phi_{2s}'}{\alpha_{2s}'}$.	$a_{2s}' - r_{2s} = \frac{r_{2s}\sin\phi_{2s}'}{\sin\alpha_{2s}'}$.
. . . .	. . . .
$\phi_{2t} = \frac{(a_{2t-2}' - r_{2t} - d_{2t-1})\,\alpha_{2t-2}'}{r_{2t}}$.	$\sin\phi_{2t} = \frac{(a_{2t-2}' - r_{2t} - d_{2t-1})\sin\alpha_{2t-2}'}{r_{2t}}$.
$\phi_{2t}' = \phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}$.	$\sin\phi_{2t}' = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}$.
$\delta_{2t} = \phi_{2t} - \phi_{2t}'$.	$\delta_{2t} = \phi_{2t} - \phi_{2t}'$.
$\alpha_{2t}' = \delta_{2t} + \alpha_{2t-2}'$.	$\alpha_{2t}' = \delta_{2t} + \alpha_{2t-2}'$.
$h_{2t} = r_{2t}(\phi_{2t}' + \alpha_{2t}')$.	$h_{2t} = r_{2t}\sin(\phi_{2t}' + \alpha_{2t}')$.
$a_{2t}' - r_{2t} = \frac{r_{2t}\phi_{2t}'}{\alpha_{2t}'}$.	$a_{2t}' - r_{2t} = \frac{r_{2t}\sin\phi_{2t}'}{\sin\alpha_{2t}'}$.

(*d*) The (s + 1)th Radius infinite, the others not very large.

Paraxial Ray.	Rim Ray.
$\eta_0 = \frac{h_0}{r_0}$.	$\sin\eta_0 = \frac{h_0}{r_0}$.
$\alpha_0 = \frac{h_0}{u}$.	$\tan\alpha_0 = \frac{h_0}{u - 2\,r_0\sin^2\frac{\eta_0}{2}}$.
$\phi_0 = \eta_0 - \alpha_0$.	$\phi_0 = \eta_0 - \alpha_0$.
$\phi_0' = \phi_0\frac{n_{-1}}{n_{+1}}$.	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}$.
$\delta_0 = \phi_0 - \phi_0'$.	$\delta_0 = \phi_0 - \phi_0'$.
$\alpha_0' = \delta_0 + \alpha_0$.	$\alpha_0' = \delta_0 + \alpha_0$.
$a_0' - r_0 = \frac{r_0\phi_0'}{\alpha_0'}$.	$a_0' - r_0 = \frac{r_0\sin\phi_0'}{\sin\alpha_0'}$.

(*d*) The $(s+1)$th Radius infinite, the others not very large—*Cont.*

Paraxial Ray.	Rim Ray.
$\phi_2 = \frac{(a_0' - r_2 - d_1)\,\alpha_0'}{r_2}.$	$\sin\phi_2 = \frac{(a_0' - r_2 - d_1)\sin\alpha_0'}{r_2}.$
$\phi_2' = \phi_2\frac{n_1}{n_3}.$	$\sin\phi_2' = \sin\phi_2\frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\alpha_2' = \delta_2 + \alpha_0'.$	$\alpha_2' = \delta_2 + \alpha_0'.$
$h_2 = r_2(\phi_2' + \alpha_2').$	$h_2 = r_2\sin(\phi_2' + \alpha_2').$
$a_2' - r_2 = \frac{r_2\phi_2'}{\alpha_2'}.$	$a_2' - r_2 = \frac{r_2\sin\phi_2'}{\sin\alpha_2'}.$
. . . . . .	. . . .
$\phi_{2s} = -\alpha'_{2s-2}.$	$\phi_{2s} = -\alpha'_{2s-2}.$
$\phi'_{2s} = \phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$	$\sin\phi'_{2s} = \sin\phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$
$\alpha'_{2s} = -\phi'_{2s}.$	$\alpha'_{2s} = -\phi'_{2s}.$
$a'_{2s} = \frac{(a'_{2s-2} - d_{2s-1})\,\alpha'_{2s-2}}{\alpha'_{2s}}.$	$a'_{2s} = \frac{(a'_{2s-2} - d_{2s-1})\tan\alpha'_{2s-2}}{\tan\alpha'_{2s}}.$
$h_{2s} = a'_{2s}\alpha'_{2s}.$	$h_{2s} = a'_{2s}\tan\alpha'_{2s}.$
. . . .	. . . .
$\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\,\alpha'_{2t-2}}{r_{2t}}.$	$\sin\phi_{2t} =$ $= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\alpha'_{2t-2}}{r_{2t}}.$
$\phi'_{2t} = \phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$	$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$
$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$	$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$
$h_{2t} = r_{2t}(\phi'_{2t} + \alpha'_{2t}).$	$h_{2t} = r_{2t}\sin(\phi'_{2t} + \alpha'_{2t}).$
$a'_{2t} - r_{2t} = \frac{r_{2t}\phi'_{2t}}{\alpha'_{2t}}.$	$a'_{2t} - r_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\alpha'_{2t}}.$

(*e*) The $(s + 1)^{th}$ Radius very large, the others not.

Paraxial Ray.	Rim Ray.
$\eta_0 = \frac{h_0}{r_0}.$	$\sin\eta_0 = \frac{h_0}{r_0}.$
$\alpha_0 = \frac{h_0}{u}.$	$\tan\alpha_0 = \frac{h_0}{u - 2r_0\sin^2\frac{\eta_0}{2}}.$
$\phi_0 = \eta_0 - \alpha_0.$	$\phi_0 = \eta_0 - \alpha_0.$
$\phi_0' = \phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \phi_0 - \phi_0'.$	$\delta_0 = \phi_0 - \phi_0'.$
$\alpha_0' = \delta_0 + \alpha_0.$	$\alpha_0' = \delta_0 + \alpha_0.$
$a_0' - r_0 = \frac{r_0\phi_0'}{\alpha_0'}.$	$a_0' - r_0 = \frac{r_0\sin\phi_0'}{\sin\alpha_0'}.$
$\phi_2 = \frac{(a_0' - r_2 - d_1)\,\alpha_0'}{r_2}$	$\sin\phi_2 = \frac{(a_0' - r_2 - d_1)\sin\alpha_0'}{r_2}.$
$\phi_2' = \phi_2\frac{n_1}{n_3}.$	$\sin\phi_2' = \sin\phi_2\frac{n_1}{n_3}.$
$\delta_2 = \phi_2 - \phi_2'.$	$\delta_2 = \phi_2 - \phi_2'.$
$\alpha_2' = \delta_2 + \alpha_0'.$	$\alpha_2' = \delta_2 + \alpha_0'.$
$h_2 = r_2(\phi_2' + \alpha_2').$	$h_2 = r_2\sin(\phi_2' + \alpha_2').$
$a_2' - r_2 = \frac{r_2\phi_2'}{\alpha_2'}.$	$a_2' - r_2 = \frac{r_2\sin\phi_2'}{\sin\alpha_2'}.$
. . . .	. . . .
. . . .	. . . .
$\phi_{2s} = \frac{(a_{2's-2} - r_{2s} - d_{2s-1})\,\alpha_{2's-2}}{r_{2s}}.$	$\sin\phi_{2s} =$ $= \frac{(a_{2's-2} - r_{2s} - d_{2s-1})\sin\alpha_{2's-2}}{r_{2s}}.$
$\phi_{2's} = \phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$	$\sin\phi_{2's} = \sin\phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$

(*e*) The $(s+1)^{\text{th}}$ Radius very large, the others not—*Cont.*

Paraxial Ray.	Rim Ray.
$\delta_{2s} = \phi_{2s} - \phi'_{2s}.$	$\delta_{2s} = \phi_{2s} - \phi'_{2s}.$
$\alpha'_{2s} = \delta_{2s} + \alpha'_{2s-2}.$	$\alpha'_{2s} = \delta_{2s} + \alpha'_{2s-2}.$
$\eta_{2s} = \phi'_{2s} + \alpha'_{2s}.$	$\eta_{2s} = \phi'_{2s} + \alpha'_{2s}.$
$h_{2s} = h_{2s-2} - d_{2s-1}\,\alpha'_{2s-2}.$	$h_{2s} = h_{2s-2} -$ $- (d_{2s-1} \mp \Delta d_{2s-2} \pm \Delta d_{2s}) \tan \alpha'_{2s-2}.$†
$a'_{2s} = \frac{h_{2s}}{\alpha'_{2s}}.$	$a'_{2s} = \frac{h_{2s}}{\tan \alpha'_{2s}} + 2\, r_{2s} \sin^2 \frac{\eta_{2s}}{2}.$
. . . .	. . . .
. . . .	. . . .
$\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\,\alpha'_{2t-2}}{r_{2t}}.$	$\sin \phi_{2t} =$ $= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1}) \sin \alpha'_{2t-2}}{r_{2t}}.$
$\phi'_{2t} = \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$	$\sin \phi'_{2t} = \sin \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$
$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$	$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$
$h_{2t} = r_{2t}(\phi'_{2t} + \alpha'_{2t}).$	$h_{2t} = r_{2t} \sin (\phi'_{2t} + \alpha'_{2t}).$
$a'_{2t} - r_{2t} = \frac{r_{2t}\phi'_{2t}}{\alpha'_{2t}}.$	$a'_{2t} - r_{2t} = \frac{r_{2t} \sin \phi'_{2t}}{\sin \alpha'_{2t}}.$

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† Where the double sign occurs the upper sign should be used when the radius having the corresponding suffix is positive, and vice versa.—Ed.

IV. FOR RAYS WHICH PROCEED FROM AN OBJECT POINT AT A FINITE DISTANCE, BUT OUT OF THE AXIS

(*a*) For oblique Rays in the axial plane.

(1) No Radius very large.	(2) The first Radius infinite.
$\sin\phi_0 = \frac{(e_{2t,0} - r_0)\sin\tau_0}{r_0}$. ‡	—
$\eta_0 = \phi_0 + \tau_0$.	—
$h_0 = r_0 \sin\eta_0$.	$h_0 = e_{2t,0}\tan\tau_0$. ‡
$\sin\eta_0 = \frac{h_0}{r_0}$.	—
$\tan\alpha_0 = \frac{h_0 + u^*}{u - 2r_0\sin^2\frac{\eta_0}{2}}$. § ¶	$\tan\alpha_0 = \frac{h_0 + u^*}{u}$. § ¶
$\phi_0 = \eta_0 - \alpha_0$.	$\phi_0 = -\alpha_0$.
$\sin\phi_0' = \sin\phi_0 \frac{n_{-1}}{n_{+1}}$.	$\sin\phi_0' = \sin\phi_0 \frac{n_{-1}}{n_{+1}}$.
$\delta_0 = \phi_0 - \phi_0'$.	$\delta_0 = \phi_0 - \phi_0'$.
$\alpha_0' = \delta_0 + \alpha_0$.	$\alpha_0' = \delta_0 + \alpha_0$.
$a_0' - r_0 = \frac{r_0\sin\phi_0'}{\sin\alpha_0'}$.	$a_0' = \frac{h_0}{\tan\alpha_0'}$.
. . . .	. . . .
. . . .	. . . .
$\sin\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\alpha'_{2t-2}}{r_{2t}}$.	$\sin\phi_{2t} = \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\alpha'_{2t-2}}{r_{2t}}$.

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‡ Formulæ for the calculation of the approximate value of h_0 for the principal ray for which $\alpha_0 = \tau_0$, $\alpha_0' = \tau_0' \ldots \alpha_{2t} = \tau_{2t}$, $\alpha_{2t}' = \tau_{2t}'$.

§ For the principal ray τ_0 is given.

¶ u^* = distance of oblique object point from the axis.—Ed.

(*a*) For oblique Rays in the axial plane—*Cont.*

(1) No Radius very large.	(2) The first Radius infinite.
$\sin \phi'_{2t} = \sin \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$	$\sin \phi'_{2t} = \sin \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}.$
$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$	$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$
$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$	$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$
$h_{2t} = r_{2t} \sin (\phi'_{2t} + \alpha'_{2t}).$	$h_{2t} = r_{2t} \sin (\phi'_{2t} + \alpha'_{2t}).$
$a'_{2t} - r_{2t} = \frac{r_{2t} \sin \phi'_{2t}}{\sin \alpha'_{2t}}.$	$a'_{2t} - r_{2t} = \frac{r_{2t} \sin \phi'_{2t}}{\sin \alpha'_{2t}}.$
$h'_{2t} = (a'_{2t} - v'_{2t}) \tan \alpha'_{2t}.$	$h'_{2t} = (a'_{2t} - v'_{2t}) \tan \alpha'_{2t}.$
$x = \frac{\sin \phi_0 r_0 \tan \alpha_0}{\sin \alpha_0}.$ ‡	$x = h_0.$§ ‡
$z^2 = x^2 + y^2.$	$z^2 = x^2 + y^2.$
$\cos \zeta_0 = \frac{x}{z}.$	$\cos \zeta_0 = \frac{x}{z}.$
$q_0 = z.$	$q_0 = z.$

‡ Formulæ for the determination of the constants for the rays not in the axial plane.

§ In this case the perpendicular plane of reference is the surface itself.—Ed.

(*a*) For oblique Rays in the axial plane—*Cont.*

(3) The first Radius very large.	(4) The $(s+1)$th Radius infinite.
$\left.\begin{aligned} \sin\phi_0 &= \frac{(e_{2t,0}-r_0)\sin\tau_0}{r_0}. \\ \eta_0 &= \phi_0+\tau_0. \\ h_0 &= r_0\sin\eta_0. \end{aligned}\right\}\ddagger$	$\left.\begin{aligned} \sin\phi_0 &= \frac{(e_{2t,0}-r_0)\sin\tau_0}{r_0}. \\ \eta_0 &= \phi_0+\tau_0. \\ h_0 &= r_0\sin\eta_0. \end{aligned}\right\}\ddagger$
$\sin\eta_0 = \frac{h_0}{r_0}.$	$\sin\eta_0 = \frac{h_0}{r_0}.$
$\tan\alpha_0 = \frac{h_0+u^*}{u-2r_0\sin^2\frac{\eta_0}{2}}.\ \S\ \P$	$\tan\alpha_0 = \frac{h_0+u^*}{u-2r_0\sin^2\frac{\eta_0}{2}}.\ \S\ \P$
$\phi_0 = \eta_0-\alpha_0.$	$\phi_0 = \eta_0-\alpha_0.$
$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$	$\sin\phi_0' = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$
$\delta_0 = \phi_0-\phi_0'.$	$\delta_0 = \phi_0-\phi_0'.$
$\alpha_0' = \delta_0+\alpha_0.$	$\alpha_0' = \delta_0+\alpha_0.$
$a_0' = \frac{h_0}{\tan\alpha_0'}+2r_0\sin^2\frac{\eta_0}{2}.$	$a_0'-r_0 = \frac{r_0\sin\phi_0'}{\sin\alpha_0'}.$
. . . .	. . . .
$\sin\phi_{2s} = \frac{(a_{2s-2}'-r_{2s}-d_{2s-1})\sin\alpha_{2s-2}'}{r_{2s}}.$	$\phi_{2s} = -\alpha_{2s-2}'.$
$\sin\phi_{2s}' = \sin\phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$	$\sin\phi_{2s}' = \sin\phi_{2s}\frac{n_{2s-1}}{n_{2s+1}}.$
$\delta_{2s} = \phi_{2s}-\phi_{2s}'.$	
$\alpha_{2s}' = \delta_{2s}+\alpha_{2s-2}'.$	$\alpha_{2s}' = -\phi_{2s}'.$
$h_{2s} = r_{2s}\sin(\phi_{2s}'+\alpha_{2s}').$	$a_{2s}' = \frac{(a_{2s-2}'-d_{2s-1})\tan\alpha_{2s-2}'}{\tan\alpha_{2s}'}.$
$a_{2s}'-r_{2s} = \frac{r_{2s}\sin\phi_{2s}'}{\sin\alpha_{2s}'}.$	$h_{2s} = a_{2s}'\tan\alpha_{2s}'.$
. . . .	. . . .

‡ Formulæ for the calculation of the approximate value of h_0 for the principal ray for which $\alpha_0 = \tau_0$, $\alpha_0' = \tau_0'$, ... $\alpha_{2t} = \tau_{2t}$, $\alpha_{2t}' = \tau_{2t}'$.

§ For the principal ray τ_0 is given.

¶ u^* = distance of oblique object point from the axis.—ED.

(*a*) For oblique Rays in the axial plane—*Cont.*

(3) The first Radius very large.

$$\sin\phi_{2t} =$$

$$= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\alpha'_{2t-2}}{r_{2t}}.$$

$$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$$

$$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$$

$$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$$

$$h_{2t} = r_{2t}\sin(\phi'_{2t} + \alpha'_{2t}).$$

$$a'_{2t} - r_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\alpha'_{2t}}.$$

$$h'_{2t} = (a'_{2t} - v'_{2t})\tan\alpha'_{2t}.$$

$$\left.\begin{aligned} x &= \frac{\sin\phi_0 r_0\tan\alpha_0}{\sin\alpha_0}. \\ z^2 &= x^2 + y^2. \\ \cos\zeta_0 &= \frac{x}{z}. \\ q_0 &= z. \end{aligned}\right\}\ddagger$$

(4) The $(s+1)$th Radius infinite.

$$\sin\phi_{2t} =$$

$$= \frac{(a'_{2t-2} - r_{2t} - d_{2t-1})\sin\alpha'_{2t-2}}{r_{2t}}.$$

$$\sin\phi'_{2t} = \sin\phi_{2t}\frac{n_{2t-1}}{n_{2t+1}}.$$

$$\delta_{2t} = \phi_{2t} - \phi'_{2t}.$$

$$\alpha'_{2t} = \delta_{2t} + \alpha'_{2t-2}.$$

$$h_{2t} = r_{2t}\sin(\phi'_{2t} + \alpha'_{2t}).$$

$$a'_{2t} - r_{2t} = \frac{r_{2t}\sin\phi'_{2t}}{\sin\alpha'_{2t}}.$$

$$h'_{2t} = (a'_{2t} - v'_{2t})\tan\alpha'_{2t}.$$

$$\left.\begin{aligned} x &= \frac{\sin\phi_0 r_0\tan\alpha_0}{\sin\alpha_0}. \\ z^2 &= x^2 + y^2. \\ \cos\zeta_0 &= \frac{x}{z}. \\ q_0 &= z. \end{aligned}\right\}\ddagger$$

(5) The $(s+1)$th Radius very large.

$$\left.\begin{aligned} \sin\phi_0 &= \frac{(e_{2t,0} - r_0)\sin\tau_0}{r_0}. \\ \eta_0 &= \phi_0 + \tau_0. \\ h_0 &= r_0\sin\eta_0. \end{aligned}\right\}\S$$

$$\sin\eta_0 = \frac{h_0}{r_0}.$$

(5) The $(s+1)$th Radius very large.

$$\tan\alpha_0 = \frac{h_0 + u^*}{u - 2r_0\sin^2\frac{\eta_0}{2}}.\ \|\ \P$$

$$\phi_0 = \eta_0 - \alpha_0.$$

$$\sin\phi'_0 = \sin\phi_0\frac{n_{-1}}{n_{+1}}.$$

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‡ Formulæ for the determination of the constants for the rays not in the axial plane.

§ Formulæ for the calculation of the approximate value of h_0 for the principal ray for which $\alpha_0 = \tau_0$, $\alpha'_0 = \tau'_0 \ldots \alpha_{2t} = \tau_{2t}$, $\alpha'_{2t} = \tau'_{2t}$.

‖ For the principal ray τ_0 is given.

¶ u^* = the distance of oblique object point from the axis.—Ed.

(a) For oblique Rays in the axial plane—*Cont.*

(5) The (s + 1)th Radius very large.

$\delta_0 = \phi_0 - \phi_0'$.

$\alpha_0' = \delta_0 + \alpha_0$.

$a_0' - r_0 = \frac{r_0 \sin \phi_0'}{\sin \alpha_0'}$.

. . . .

. . . .

$\sin \phi_{2s} =$

$$= \frac{(a_{2s-2}' - r_{2s} - d_{2s-1}) \sin \alpha_{2s-2}'}{r_{2s}}.$$

$\sin \phi_{2s}' = \sin \phi_{2s} \frac{n_{2s-1}}{n_{2s+1}}$.

$\delta_{2s} = \phi_{2s} - \phi_{2s}'$.

$\alpha_{2s}' = \delta_{2s} + \alpha_{2s-2}'$.

$\eta_{2s} = \phi_{2s}' + \alpha_{2s}'$.

$h_{2s} = h_{2s-2} - (d_{2s-1} \mp$
$\mp \Delta d_{2s-2} \pm \Delta d_{2s}) \tan \alpha_{2s-2}'$. ‡

$a_{2s}' = \frac{h_{2s}}{\tan \alpha_{2s}'} + 2 r_{2s} \sin^2 \frac{\eta_{2s}}{2}$.

. . . .

. . . .

(5) The (s + 1)th Radius very large.

$\sin \phi_{2t} =$

$$= \frac{(a_{2t-2}' - r_{2t} - d_{2t-1}) \sin \alpha_{2t-2}'}{r_{2t}}.$$

$\sin \phi_{2t}' = \sin \phi_{2t} \frac{n_{2t-1}}{n_{2t+1}}$.

$\delta_{2t} = \phi_{2t} - \phi_{2t}'$.

$\alpha_{2t}' = \delta_{2t} + \alpha_{2t-2}'$.

$h_{2t} = r_{2t} \sin (\phi_{2t}' + \alpha_{2t}')$.

$a_{2t}' - r_{2t} = \frac{r_{2t} \sin \phi_{2t}'}{\sin \alpha_{2t}'}$.

$h_{2t}' = (a_{2t}' - v_{2t}') \tan \alpha_{2t}'$.

$$\left.\begin{aligned} x &= \frac{\sin \phi_0 r_0 \tan \alpha_0}{\sin \alpha_0}. \\ z^2 &= x^2 + y^2. \\ \cos \zeta_0 &= \frac{x}{z}. \\ q_0 &= z. \end{aligned}\right\} \S$$

---

‡ Where the double sign occurs the upper sign should be used when the radius having the corresponding suffix is positive, and vice versa.—**Ed.**

§ Formulæ for the determination of the constants for the rays not in the axial plane.

FORMULÆ FOR THE DETERMINATION OF THE RADII OF A LENS

I. A SINGLE LENS

(*a*) General Formulæ

Known data: h_0, δ_0, δ_2, d_1, n_{+1} and $n_{-1} = n_3 = 1$.

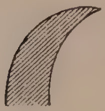
To be found: r_0 and r_2.

[*To p. 41.*

(*b*) Formulæ for Special Cases

Convergent Lenses: $\beta_{0,2} = +1$.

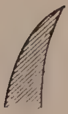
1. $\delta_0 = +2$; $\delta_2 = -1$; $\beta_{0,2} = \delta_0 + \delta_2 = +1$.



$$r_0 = \frac{h_0(n_{+1} - 1)}{2n_{+1}},$$

$$r_2 = \frac{(h_0 - 2d_1)(n_{+1} - 1)}{2n_{+1} - 1}.$$

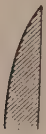
2. $\delta_0 = +1$; $\delta_2 = 0$; $\beta_{0,2} = \delta_0 + \delta_2 = +1$.



$$r_0 = \frac{h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = (h_0 - d_1).$$

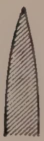
3. $\delta_0 = \frac{1}{n_{+1}}$; $\delta_2 = \frac{n_{+1} - 1}{n_{+1}}$; $\beta_{0,2} = \delta_0 + \delta_2 = +1$.



$$r_0 = h_0(n_{+1} - 1),$$

$$r_2 = \infty.$$

4. $\delta_0 = \delta_2 = 0{\cdot}5$; $\beta_{0,2} = \delta_0 + \delta_2 = +1$.



$$r_0 = \frac{2h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = -\frac{(2h_0 - d_1)(n_{+1} - 1)}{2 - n_{+1}}.$$

[*Continued on p. 42.*

r_0 and r_2 are found in terms of the focal length as unity.

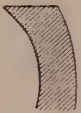
$$r_0 = \frac{h_0(n_{+1} - 1)}{\delta_0 n_{+1}},$$

$$r_2 = \frac{(h_0 - d_1\delta_0)(n_{+1} - 1)}{-\delta_2 + \delta_0(n_{+1} - 1)}.$$

(*b*) Formulæ for Special Cases

Divergent Lenses: $\beta_{0,2} = -1$.

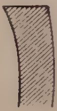
1. $\delta_0 = -2;\ \delta_2 = +1;\ \beta_{0,2} = \delta_0 + \delta_2 = -1.$



$$r_0 = -\frac{h_0(n_{+1} - 1)}{2n_{+1}},$$

$$r_2 = -\frac{(h_0 + 2d_1)(n_{+1} - 1)}{2n_{+1} - 1}.$$

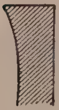
2. $\delta_0 = -1;\ \delta_2 = 0;\ \beta_{0,2} = \delta_0 + \delta_2 = -1.$



$$r_0 = -\frac{h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = -(h_0 + d_1).$$

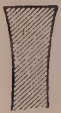
3. $\delta_0 = -\frac{1}{n_{+1}};\ \delta_2 = -\frac{n_{+1} - 1}{n_{+1}};\ \beta_{0,2} = \delta_0 + \delta_2 = -1.$



$$r_0 = -h_0(n_{+1} - 1),$$

$$r_2 = \infty.$$

4. $\delta_0 = \delta_2 = -0{\cdot}5;\ \beta_{0,2} = \delta_0 + \delta_2 = -1.$



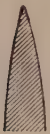
$$r_0 = -\frac{2h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = +\frac{(2h_0 + d_1)(n_{+1} - 1)}{2 - n_{+1}}.$$

[*Continued on p. 43.*

Convergent Lenses: $\beta_{0,2} = +1$ —*Cont.*

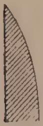
5. $\delta_0 + \delta_2 = \beta_{0,2} = +1; \; r_0 = -r_2.$



$$r_0 = -r_2 = h_0(n_{+1} - 1) \pm$$

$$\pm\sqrt{-\frac{h_0 d_1 (n_{+1} - 1)^2}{n_{+1}} + h_0^2 (n_{+1} - 1)^2}.$$

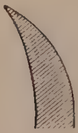
6. $\delta_0 = 0; \; \delta_2 = +1; \; \beta_{0,2} = \delta_0 + \delta_2 = +1.$



$$r_0 = \infty,$$

$$r_2 = -h_0(n_{+1} - 1).$$

7. $\delta_0 = -1; \; \delta_2 = +2; \; \beta_{0,2} = \delta_0 + \delta_2 = +1$



$$r_0 = -\frac{h_0(n_{+1} - 1)}{n_{+1}},$$

$$r_2 = -\frac{(h_0 + d_1)(n_{+1} - 1)}{n_{+1} + 1}.$$

II. A SYSTEM OF LENSES

(In the course of a trigonometrical computation)

1. To find the first radius r_0.

Given: $\delta_0, h_0, d_1, n_{+1}$ and $n_{-1} = 1$.

Then: $r_0 = \dfrac{h_0(n_{+1} - 1)}{\delta_0 n_{+1}}$.

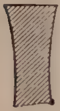
2. To find any radius, for example the third r_4.

Given: $\delta_4, a_2', \beta_{0,2}, d_3, n_3$, and n_5.

Then: $\phi_4 = \dfrac{\delta_4}{1 - \dfrac{n_3}{n_5}}$,

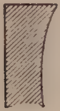
Divergent Lenses: $\beta_{0,2} = -1$ —*Cont.*

5. $\delta_0 + \delta_2 = \beta_{0,2} = -1$; $r_0 = -r_2$.



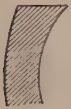
$$r_0 = -r_2 = -h_0 (n_{+1} - 1) \pm$$
$$\pm \sqrt{+\frac{h_0 d_1 (n_{+1} - 1)^2}{n_{+1}} + h_0^2 (n_{+1} - 1)^2}.$$

6. $\delta_0 = 0$; $\delta_2 = -1$; $\beta_{0,2} = \delta_0 + \delta_2 = -1$.



$$r_0 = \infty,$$
$$r_2 = +h_0 (n_{+1} - 1).$$

7 $\delta_0 = +1$; $\delta_2 = -2$; $\beta_{0,2} = \delta_0 + \delta_2 = -1$.



$$r_0 = +\frac{h_0 (n_{+1} - 1)}{n_{+1}},$$
$$r_2 = +\frac{(h_0 - d_1)(n_{+1} - 1)}{n_{+1} + 1}.$$

2. *Cont.*

and $$r_4 = \frac{(a_2' - d_3)\,\beta_{0,2}}{\beta_{0,2} + \phi_4}.$$

In certain special cases a rim ray may be under consideration without reference to the axial ray. In such a case the formulæ are as follows:—

$$\tan \phi_4 = \frac{\sin \delta_4}{\cos \delta_4 - \frac{n_3}{n_5}},$$

and $$r_4 = \frac{(a_2' - d_3) \sin \beta_{0,2}}{\sin \beta_{0,2} + \sin \phi_4}.$$

FORMULÆ FOR THE DETERMINATION OF THE MAXIMUM APERTURE OF A LENS

1. Convergent lenses:

(*a*) If neither r_0 nor r_2 is infinite, the aperture is given by:

$$h_0 = \frac{2}{r_2 - r_0 + d_1} \times$$

$$\sqrt{\frac{d_1}{2}\left(-r_0 + r_2 + \frac{d_1}{2}\right)\left(r_0 - \frac{d_1}{2}\right)\left(r_2 + \frac{d_1}{2}\right)}.$$

(*b*) If r_2 is infinite:

$$h_0 = \sqrt{2r_0d_1 - d_1^2}.$$

(*c*) If r_0 is infinite:

$$h_0 = \sqrt{-2r_2d_1 - d_1^2}.$$

2. Divergent lenses:

The maximum semi-aperture of a divergent lens is always identical with the smaller radius of curvature.

CHAPTER I

Determination of Refractive Indexes

Apparatus and Method of Measurement

In the formulæ, Volume I, pages 3 and 4, derived from the law of refraction, there are involved constants, n_{-1}, n_{+1}, n_{+3} . . ., the values of which depend upon the natures of the several refracting media. For the numerical evaluation of these formulæ it is necessary to know how to determine the refractive indexes.

It was Fraunhofer in particular who first made it possible to observe with accuracy these constants, through his discovery of a series of dark lines (Fraunhofer lines) in the solar spectrum formed by a prism. Individual lines were indicated by the capital letters, A, B, . . . H, the refractive indexes for the various colours corresponding with these lines being expressed numerically.[‡]

In Plate I (facing page 44) there is represented the solar spectrum, showing the black lines discovered by Fraunhofer. The lines are denoted by the letters proposed by him. The diagram in question is reproduced from Fraunhofer's original work.[§]

The measuring apparatus used by Fraunhofer was not of a convenient type. It was only as a result of improvements by

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[‡] "The Determination of the Refraction and the Dispersive Powers of various types of Glasses with reference to the perfecting of Achromatic Telescopes", by J. Fraunhofer (*Memoirs of the Munich Academy* for 1814 and 1815, Vol. V, 1817).

[§] The diagram is mainly of historical interest, as accuracy of detail has been lost in the double reproduction.—Ed.

C. Steinheil and Seidel,[‡] Meyerstein, and others,[§] that the apparatus became of general use. More recently simpler instruments for special purposes have been introduced by Abbe.[||]

It will be sufficient to describe the method of determining refractive indexes by means of one instrument, and for this purpose there will be described a goniometer, indicated in figs. 11 to 15 (Plate, p. 50), and constructed by C. A. Steinheil & Sons for the Optical Testing Laboratory of the Imperial Technical High School of Munich.

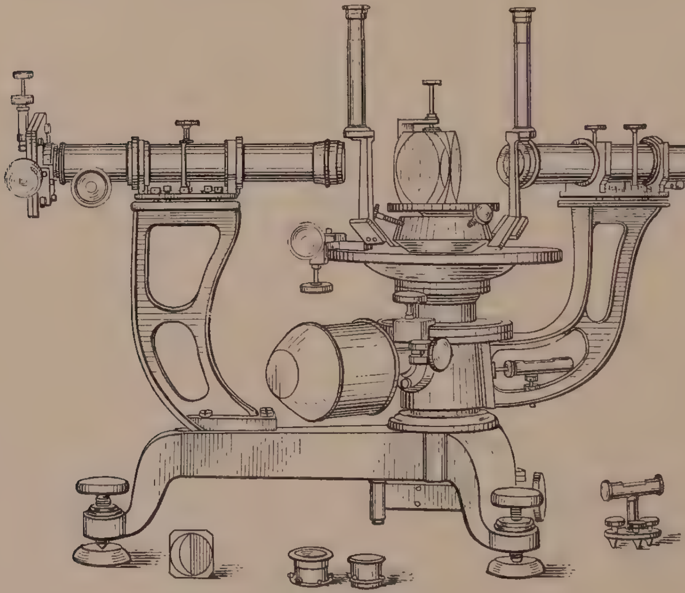

Fig. 11

The goniometer is supported upon a cast-iron base having three feet provided with levelling screws, *a*, *a* (fig. 12). The distance from the levelling screws of two of the feet, A A, to the axis of the instrument is 10·6 cm. (4·2 in.), whereas the distance in the case of the third foot, B, is 27·4 cm. (10·8 in.).

[‡] "On the Determination of the Refraction and the Dispersive Powers of various Media", by Steinheil and Seidel (*Transactions, Class II*, of the Imperial Bavarian Academy of Science, Vol. V, Part II, p. 1).

[§] "On Spectrum Apparatus", by E. Voit (Carl's *Repertory*, Vol. I, p. 93 *et seq.*).

[||] "New Apparatus for the Determination of the Refraction and the Dispersive Powers of Solid and Liquid Bodies", by E. Abbe, Jena, 1874.

The latter serves to support the collimating telescope P. Into the base there is screwed a hollow steel column, C, turned to a conical form externally. An index plate, D, is rigidly secured to the upper end of this column, and upon the upper end of a brass spindle, E, mounted within the bore of the steel column C, there is attached a holder, R, for the specimen table. On the lower end of the spindle there is provided a fine adjustment and clamping attachment. Upon the lower half of the conical column there is mounted a bronze sleeve, F, to which are attached the supporting bracket, H, of the observing telescope, O, and a balance weight, G. On the upper half of the column there is mounted a second bronze sleeve, I, which bears upon the lower portion, F, and which is provided at its lower end with a clamping plate, K, and at its upper end with a dial plate, L, having divisions engraved upon an inlaid silver ring. This portion can be clamped and finely adjusted in rotation with respect to the lower portion carrying the observing telescope. The clamping is done by means of the screw *b*, and the adjustment by means of the micrometer screw *c*. The sleeve, I, may also be clamped and finely adjusted by means of the clamp *d*, and the micrometer screw *e*, with respect to the index plate, D, which is provided with four verniers spaced 90° apart.

The table, Q, is screwed to a conical steel base having a lower spherical surface, which rests upon a corresponding spherical surface of a brass table holder, R, of dished form. The centre of the spherical surface is at the point of intersection, S, of the vertical axis of rotation of the instrument with the line of the telescope axes. The table holder, secured to the spindle E, is provided with a rim carrying three adjusting screws, *f*, spaced 120° apart and bearing upon the conical surface of the table base, which can be rotated by this means about the centre, S. A clamping spring, *g*, serves to hold the prism firmly in place on the table. To the lower end of the central spindle there is attached a gun-metal worm-wheel, T, associated with a worm which is held in gear by a spring. By means of the worm-operating head *h* (fig. 14) the table can be adjusted as desired for the setting of minimum deviation. The worm can be moved out of gear with the worm-wheel against the action of the spring, thus permitting the table to be turned freely by hand.

The graduated circle, which has a diameter of about 16 cm. (6·3 in.), is divided in steps of 10 minutes. There are provided verniers, 60 divisions of which correspond with 59 divisions of the circle. Direct readings of 10 seconds of arc can thus be obtained. By means of two reading microscopes, M, M, movable about the axis of the instrument, readings can be taken, across two diameters, of the indications of verniers spaced 180° apart. The reading microscopes are so supported that their free rotation about the axis of the instrument is unhindered.

Both the collimating and observing telescopes have objectives of 2·7 cm. (1·06 in.) aperture and 24·4 cm. (9·6 in.) focal length. The telescopes are provided with eyepiece focusing heads, *j*. They are supported upon journals within bearings, K, in which they are clamped by means of the upper halves. The collimator, the supporting bracket of which is carried upon the limb B, has at its image end a Kruss slit of the double-slide type (fig. 13). Upon the slide plate there is situated a plane parallel glass, *l*, having two parallel lines engraved upon its face. The plane containing the two parallel lines lies in the focus of the objective, and the centre of the square, formed by the parallel lines and the edges of the slit, lies on the optical axis of the telescope. The focusing of the lines is effected by means of the knurled head *j*, and the position of the glass plate is adjusted by the head *m*. The observing telescope, which is carried within bearings on the arm H, is provided with a micrometer eyepiece having an adjustable graticule.

Observations with this instrument, involved in the determination of refractive indexes of a particular type of glass, comprise the adjustment of the apparatus; the determination of the refracting angle of the glass in its prismatic form; and the measurement of the minimum deviation for definite colours of the beam of light that passes through the glass. The instrument also serves for the determination of the ratios of the medium dispersions of two types of glasses.

Adjustment of the Apparatus

1. The observing telescope, O, must be so focused that the observer can see a distant object clearly defined and without parallax relatively to the cross-wires.

For this purpose the eyepiece is focused on the cross-wires; the telescope is then directed towards a distant object, such as a star, and the eyepiece tube carrying both the eyepiece and the cross-wires is moved to or from the objective until the image is sharply defined. When the focus is correct, there should be no relative movement between the focused image and the cross-wires when the observer's eye is moved towards the right or left in front of the eyepiece; that is, there must be no parallax between the image and the cross-wires. At the same time, when the telescope is rotated in its bearings, the point of intersection of the cross-wires must cover the same part of the object; that is, the cross-wires must be centred.

The distance of the eyepiece from the cross-wires depends upon the eyesight of the observer. If the instrument is to be used by several observers, it is advisable that the eyepiece should be adjusted for a normal eye, and that the eyesight of the various observers should be corrected by means of glasses that will make their eyesight normal. For a short-sighted eye there should be selected the weakest negative spectacle lens, and for a long-sighted eye the strongest positive lens that will impart to the eye in question its full acuity of vision for distant objects.

More accurate adjustment of the observing telescope for infinite focus is obtainable by the use of two additional telescopes. All three telescopes should first be so adjusted, by the method described above, until the image of a distant object is sharply defined and exhibits no parallax. Then the several telescopes should be compared with one another in pairs, the one telescope being used to observe the image of the cross-wires of the other telescope. The focusing is correct when in all cases the cross-wires of the telescopes appear sharply defined and without parallax. The correction to be applied to each telescope should be one-quarter of the total apparent error.

2. The slit must be set in the focal plane of the objective of the collimator, P.

While viewing the slit through the collimator objective, by means of the observing telescope, the slit should be racked in or out by means of the adjusting head, until it is clearly defined without parallax in the focal plane of the observing telescope.

3. The plane of the graduated circle must be horizontal, and the axis of rotation of the instrument must be vertical.

Within a recess of the arm, H, which carries the observing telescope there is mounted a level, *o*. The arm should be placed in the direction of one of the base screws and levelled by means of it. If, then, the arm is rotated through 180° it may be found that it is no longer level. The necessary correction should be effected partly by means of the levelling screw in question and partly by means of the correction screw on the level itself. This process should be repeated until the level is adjusted and the axis of rotation of the instrument lies in a vertical plane containing the levelling screw in question. The arm carrying the level should then be rotated through 90°, and any error, as indicated by the level, corrected with respect to the two other base screws, which should be rotated by equal amounts in opposite directions. In this way the axis of rotation is brought into a vertical plane normal to the previous one, and is thus set truly vertical. Both the graduated circle and index must accordingly lie horizontally, since the latter is attached normally to the axis of rotation, and the graduated circle lies in its plane.

4. The telescope axes must be set horizontally, and must lie in the same plane.

To effect this adjustment, the observing telescope must be levelled by a suitable adjustment of the bearings in accordance with the indications of a spirit-level mounted above the telescope. The collimator must be similarly levelled. Further, when the collimator is rotated about its axis, the collimator slit must remain in the same position relatively to the axis. The necessary correction is made by displacing the glass plate *l*. It then remains to be seen whether or not the image of this fixed point coincides with the point of intersection of the cross-wires of the observing telescope. By means of the pulling and pushing screws on the carrier of the observing telescope the axis of the latter can be raised or lowered until the points in question coincide. The axis of the observing telescope must at the same time be kept horizontal.

5. The prism table must lie horizontally.

To effect this adjustment, a small level having three adjustable base screws should be placed on the table. Partly by means of these base screws and partly by means of the adjusting screws *f*, the table must be so adjusted that when it is rotated the bubble of the spirit-level remains stationary.

Determination of the Prism Angle

1. In the first instance the prism should be examined as regards the optical flatness of the surfaces and the verticality of the prism refracting edge relatively to its base, to ensure that it is suitable for the purpose in question. It should be placed upon the table with its refracting edge directed towards the objective of the collimating telescope, and in such a position that the light falls symmetrically upon both prism faces, so as to fill them equally with respect to the eye, in order that the illumination of the two reflected images may be approximately equal. To test the flatness of the prism surfaces, the reflected image of the slit in both the horizontal and vertical positions should be examined without displacing the eyepiece of the observing telescope. Both images should be clearly defined and undistorted.

If accurate plane test-plates are available, it is easier to examine the figure of the prism surfaces by means of them.

Observations should then be made as to whether or not the image of the middle of the slit, when reflected from each of the two prism surfaces, coincides with the point of intersection of the cross-wires of the observing telescope. This will occur if the base of the prism is ground normally to the refracting edge.

If this is not the case, the prism table must be adjusted by means of the screws *f* until the refracting edge of the prism is vertical. When this is done, the table will no longer be horizontal. To ensure that the prism edge will remain vertical during the subsequent adjustment of the table, the prism should be clamped to the table by means of the spring *g*.

2. The angle of the prism is determined on the goniometer by measuring the angle between the images of the slit reflected from the two prism surfaces. The value of the prism angle is then obtained by dividing the observed angle by two.

As before, the edge of the prism should be directed towards the objective of the collimator, and the light should fall symmetrically upon the two prism faces, but, in addition, the prism should be so situated upon the table that the observing telescope, when set to view the reflected image of the slit, is directed towards the middle of the prism surface. The slit should be made somewhat narrow, about 0·5 mm.; the clamp *d* should be bound, and *b*, on the other hand, loosened. With the prism so arranged, readings should be taken on two of the verniers, spaced 180° apart, say, I and III. On the vernier I complete readings should be taken, and on III only the minutes and seconds. Then the observing telescope should be directed so as to view the image of the slit reflected from the first prism face. When this is done, the clamp *b* should be bound and the setting finely adjusted by means of the micrometer screw *c*. In order to observe the reflected image from the second surface, clamp *d* should be loosened, and the telescope and the graduated circle rotated until the image appears in the field of view. Having observed whether the measurement is in the direction with or against the graduations, the clamp *d* should be bound and the setting finely adjusted by means of *e*. Then the verniers I and III should again be read. Should the observing telescope be in a position which makes it awkward to read the scale, the clamp *b* may be loosened and the telescope displaced before the reading is taken.

On the completion of these measurements the first reading should be deducted from the second, but if the second is numerically smaller, 360° should be added to it. Half of the angle thus obtained is the angle of the prism (fig. 16). The proof of this statement may be obtained from any handbook of physics.‡ When the direction of the measurement is opposite to that of the graduations, the second reading should be deducted from the first, 360° being added when necessary.

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‡ As, for example, *Light for Students*, by Edwin Edser.—Ed.

Repeated observations may be made in order to obtain more accurate results. To effect this the clamp *b* should be loosened after the first measurement has been completed; the reflected image from the first surface should again be brought into the field of view; the clamp *b* secured; and the fine adjustment made by means of *c*. Then *d* should be loosened; the telescope rotated, together with the graduated circle, until the image of the slit reflected from the second surface appears in the field of view; *d* should then again be clamped, and the fine adjustment made by means of *e*.

If two observations are considered sufficient, the readings should then be taken. But if a larger number is required, the operations described above should be repeated. It will be understood that the number of the measurements taken should be noted, and that the measurements should all be in the same sense, that is to say, all in the direction of the graduations or all against it. The number of repetitions depends upon the angle to be measured. It is best chosen so that the readings on the graduated circle always fall on different parts of the scale. In this way small graduation errors are as far as possible eliminated. For example, if, as is generally the case, the prism angle were about 60°, and if the observations were repeated three times, then the repeated readings would have to be made on approximately the same position of the graduated circle. In a single observation, if the first reading were zero, the second reading would be 120° and the third 240°, and only the fourth would again be zero. Therefore, in the case of a 60° prism, the observations should be repeated four times. The required result is found when the sum of the observations is divided by that number, which is obtained when the number of measurements is multiplied by two, care being taken to note whether and how often 360° has to be added.

Determination of the Angle of Deviation

1. The prism should be examined as regards the homogeneity of the glass.

The collimator should be so directed that the slit is illuminated by the sun. The slit should be made narrower, about 0·1 mm., and the spectrum formed by the prism should be examined for minimum deviation after the prism has been adjusted, as described later. If the glass is homogeneous, the Fraunhofer lines will be clearly defined, and the angle of deviation between each pair of Fraunhofer lines will have the same value when viewed through different parts of the prism.

2. The prism should be so adjusted upon the table that the illumination of the spectrum produced is a maximum.

To effect this, minimum deviation for the Fraunhofer line F should be approximately adjusted, as described below, and it should be observed whether or not the axis of the observing telescope is directed towards the middle of the prism surface facing it. Any necessary corrections are effected by displacing the prism upon the table. After these adjustments the prism should again be clamped to the table by means of the spring *g*, and the verticality of the refracting edge should be examined.

3. Before each determination of the angle of deviation for any particular colour, the prism table should be so adjusted that there is minimum deviation for the Fraunhofer line selected.

The clamp *b* should be loosened and the observing telescope set opposite the collimator, where it will not obstruct the view. In the first instance, suppose the prism table is so set that the unpolished surface of the prism lies parallel to the axis of the collimator. Then, after disengaging the worm T, the table should be rotated freely by hand until the unpolished surface $a\,b$, fig. 17, is brought into the new position $a_1 b_1$, thus

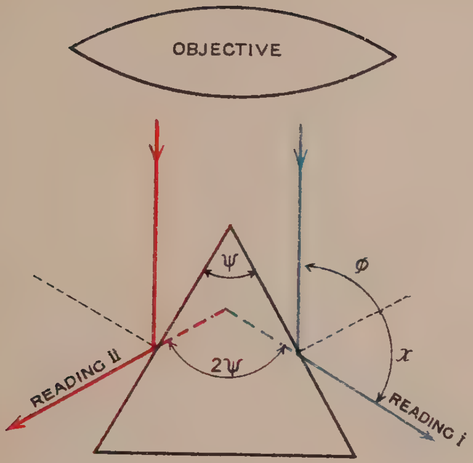

Fig. 16

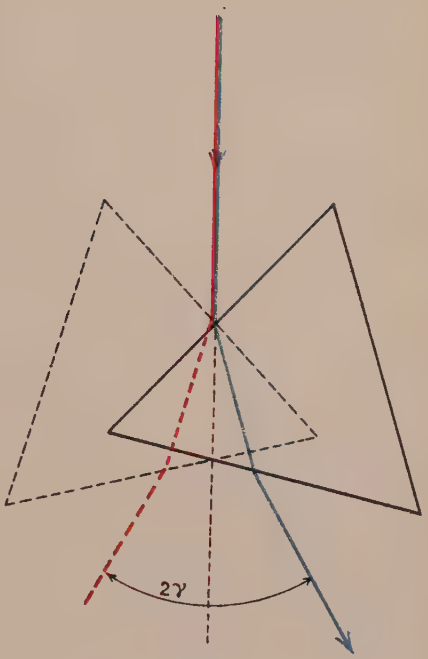

Fig. 18

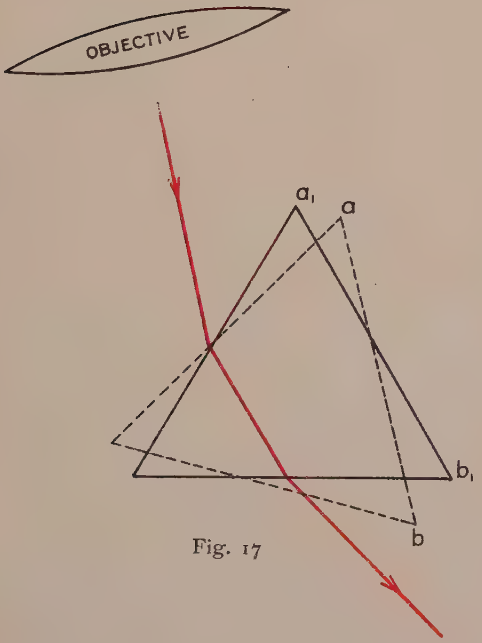

Fig. 17

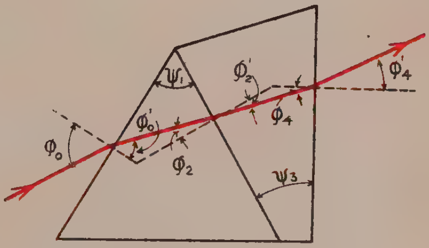

Fig. 19

[*Facing p. 54.*

enabling the eye placed behind the second refracting surface to observe the spectrum formed by the prism. During the rotation the spectrum will appear to move first in the same direction as the table and then in the opposite direction. The point at which the motion is reversed is the correct position of the prism table; the worm T should then be re-engaged.

Having first adjusted minimum deviation by the unaided eye, it is then necessary, by means of the observing telescope, to perform the adjustment exactly for the particular line for which the measurement is to be made. The prism table, Q, should be rotated by means of the head *h* until the required Fraunhofer line comes to the reversal point. The observing telescope should then be set upon the line by means of the micrometer screw *c*.

4. The angle under which the Fraunhofer line appears should be measured with the light incident, first upon the one face of the prism and then upon the other. In both cases the position of the prism should be so chosen as to give minimum deviation for the Fraunhofer line in question.

After the reading has been taken of the chosen Fraunhofer line, for example the line D, set up in the manner (fig. 18) described in Section 3, the prism table should be set free and brought into such a position that the second prism surface is directed towards the collimator. The clamp *d* having been loosened, both telescope and graduated circle should be rotated until the spectrum appears in the field of view of the observing telescope. Clamp *d* should then be secured, and the fine minimum adjustment on the line D effected by means of the slow motion of the table operated by the micrometer screw *e*. Then the reading should be taken for the second time. The difference of the two readings divided by two gives the angle of deviation, γ. Repeated observations may be made in the same manner.

5. The ratio of the refractive indexes of glass and air for a particular Fraunhofer line should be calculated as follows:—

After the prism angle, ψ, and the angle, γ, at minimum deviation, for the particular Fraunhofer line have been found in the manner described above, the ratio of the refractive indexes for glass and air (that is, the refraction ratio for air and glass) can be calculated from the formula

$$(1)\qquad \frac{n_{2s+1}}{n_{2s-1}} = \frac{\sin \frac{\gamma + \psi}{2}}{\sin \frac{\psi}{2}}.$$

In accordance with Seidel's convention, the suffixes of the term $\frac{n_{2s+1}}{n_{2s-1}}$ are such that all values associated with the refracting surfaces have even suffixes, while those associated with the intervening media are denoted by odd suffixes. The first medium has the index -1, and the first refracting surface the index 0.‡ Proof of Formula (1) is obtainable from most books of Physics.§

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‡ See Introduction, p. 1.

§ *Light for Students*, by Edwin Edser.—Ed.

A. TEST PRISM OF JENA FLINT GLASS, No. 318

I. PRISM ANGLE

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Prism Angle ×8 = 8ψ.	Prism Angle = ψ.	Error.	Error Squared.
Setting	4° 6′ 40″ 7 30	4° 7′ 5″				
			479° 31′ 0″	59° 56′ 22·5″	+0·12	0·01
4 times *with* the graduations	123 38 10 8 0	(483) 123 38 5				
			479 30 5	59 56 15·6	−6·78	45·97
4 times *with* the graduations	243 7 40 8 40	(603) 243 8 10				
			479 31 20	59 56 25·0	+2·62	6·86
4 times *with* the graduations	2 39 10 9 50	(722) 2 39 30				
			479 30 55	59 56 21·9	−0·48	0·23
4 times *with* the graduations	122 10 20 30	(482) 122 10 25				
			479 31 35	59 56 26·9	+4·52	20·43
4 times *with* the graduations	241 41 50 2 10	(601) 241 42 0		$\frac{111{\cdot}9}{5} = 22{\cdot}38$		73·50

$$\sqrt{\frac{73{\cdot}50}{(5-1)5}} = \sqrt{3{\cdot}67} = \pm 1{\cdot}91$$

$$\psi = 59^\circ\ 56'\ 22{\cdot}4'' \pm 1{\cdot}9''.$$

A. TEST PRISM OF FLINT, No. 318.—II. REFRACTION FOR LINE D*

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation ×6 = 6γ.	Angle of Deviation = γ.	Error.	Error Squared.
Setting	70° 33′ 50″ 3 50	70° 33′ 50″				
			279° 6′ 30″	46° 31′ 5·0″	+0·85	0·72
3 times *with* the graduations	349 39 50 40 50	349 40 20				
New setting†	268 48 30 9 20	268 48 55				
			279 7 0	46 31 10·0	+5·85	34·22
3 times *with* the graduations	187 55 50 6 0	(547) 187 55 55				
New setting‡	175 55 30 5 30	175 55 30				
			279 6 35	46 31 5·8	+1·65	2·72
3 times *with* the graduations	95 2 0 2 10	(455) 95 2 5				
			279 6 5	46 31 0·8	−3·35	11·22
3 times *with* the graduations	14 7 50 8 30	(374) 14 8 10				
			279 5 50	46 30 58·3	−5·85	34·22
3 times *with* the graduations	293 13 30 4 30	293 14 0				
			279 6 30	46 31 5·0	+0·85	0·72
3 times *with* the graduations	212 20 20 0 40	(572) 212 20 30				
				$\frac{24{\cdot}9}{6} = 4{\cdot}15$		83·82

$$_{D}\gamma = 46^\circ\ 31'\ 4{\cdot}15'' \pm 1{\cdot}7''. \qquad \sqrt{\frac{83{\cdot}82}{(6-1)6}} = \sqrt{2{\cdot}79} = \pm 1{\cdot}67.$$

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* The measurement is made on that one of the two D lines which lies towards the green side.

† The observations were interrupted, and therefore a new setting was required.

‡ A wrong micrometer screw was rotated, necessitating a new setting

A. TEST PRISM OF FLINT, No. 318

III. REFRACTION FOR LINE F

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation ×6 = 6γ.	Angle of Deviation = γ.	Error.	Error Squared.
Setting	187° 55′ 50″ 6 0	187° 55′ 55″				
			285° 35′ 40″	47° 35′ 56·7″	−2·5	6·25
3 times *with* the graduations	113 31 30 1 40	(473) 113 31 35				
			285 36 40	47 36 6·7	+7·5	56·25
3 times *with* the graduations	39 8 0 8 30	(399) 39 8 15				
			285 35 55	47 35 59·2	0·0	0·00
3 times *with* the graduations	324 43 40 4 40	324 44 10				
			285 35 10	47 35 51·7	−7·5	56·25
3 times *with* the graduations	250 19 0 9 40	(610) 250 19 20				
			285 36 10	47 36 1·7	+2·5	6·25
3 times *with* the graduations	175 55 30 5 30	(535) 175 55 30		$\frac{296\cdot0}{5} = 59\cdot2$		125·00

$$\sqrt{\frac{125}{(5-1)5}} = \sqrt{6\cdot25} = \pm 2\cdot5$$

$${}_{F}\gamma = 47^{\circ}\ 35'\ 59\cdot2'' \pm 2\cdot5''.$$

A. TEST PRISM OF FLINT, No. 318

IV. REFRACTION FOR LINE G_1*

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation ×6 = 6γ.	Angle of Deviation = γ.	Error.	Error Squared.
Setting	356° 22′ 30″ 3 20	356° 22′ 55″				
			292° 44′ 10″	48° 47′ 21·7″	−7·9	62·41
3 times *with* the graduations	289 6 40 7 30	(649) 289 7 5				
New setting	3 49 40 50 20	3 50 0				
			292 45 35	48 47 35·8	+6·2	38·44
3 times *with* the graduations	296 35 0 6 10	296 35 35				
			292 45 5	48 47 30·8	+1·2	1·44
3 times *with* the graduations	229 20 20 1 0	(589) 229 20 40				
			292 44 25	48 47 24·2	−5·4	29·16
3 times *with* the graduations	162 5 0 5 10	162 5 5				
			292 45 35	48 47 35·8	+6·2	38·44
3 times *with* the graduations	94 50 40 0 40	(454) 94 50 40		$\frac{148 \cdot 3}{5}$ = 29·6		169·89

$${}_{G_1}\dot{\gamma} = 48° \; 47' \; 29{\cdot}6'' \pm 2{\cdot}9''. \qquad \sqrt{\frac{169{\cdot}89}{(5-1)5}} = \sqrt{8{\cdot}\bar{49}} = \pm 2{\cdot}91$$

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* This line does not coincide with the Fraunhofer G line.

A. TEST PRISM OF FLINT, No. 318

V. REFRACTION FOR LINE H

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation ×6 = 6γ.	Angle of Deviation = γ.	Error.	Error Squared.
Setting	212° 20′ 20″ 0 40	212° 20′ 30″				
			297° 0′ 30″	49° 30′ 5·0″	−0·66	0·44
3 times *with* the graduations	149 21 0 1 0	(509) 149 21 0				
			297 1 5	49 30 10·8	+5·14	26·42
3 times *with* the graduations	86 21 50 2 20	(446) 86 22 5				
New setting*	185 21 40 1 40	185 21 40				
			296 59 50	49 29 58·3	−7·36	54·17
3 times *with* the graduations	122 21 40 1 20	(482) 122 21 30				
			297 0 40	49 30 6·7	+1·04	1·08
3 times *with* the graduations	59 22 0 2 20	(419) 59 22 10				
			297 0 45	49 30 7·5	+1·84	3·38
3 times *with* the graduations	356 22 30 3 20	356 22 55				
				$\frac{28\cdot3}{5} = 5\cdot66.$		85·49

$${}_{H}\gamma = 49^\circ\ 30'\ 5\cdot7'' \pm 2\cdot1''.$$

$$\sqrt{\frac{85\cdot49}{(5-1)5}} = \sqrt{4\cdot27} = \pm 2\cdot07.$$

* The measurement was interrupted, necessitating a new setting.

B. TEST PRISM OF JENA CROWN GLASS, No. 506

I. PRISM ANGLE

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Prism Angle ×8 = 8ψ.	Prism Angle = ψ.	Error.	Error Squared.
Setting	49° 7′ 30″ 8 0	49° 7′ 45″				
			480° 17′ 30″	60° 2′ 11·3″	+0·54	0·29
4 times *with* the graduations	169 25 20 10	(529) 169 25 15				
			480 17 20	60 2 10·0	−0·76	0·58
4 times *with* the graduations	289 42 10 3 0	(649) 289 42 35				
			480 17 40	60 2 12·5	+1·74	3·03
4 times *with* the graduations	50 0 0 0 30	(410) 50 0 15				
			480 17 15	60 2 9·4	−1·36	1·85
4 times *with* the graduations	170 17 30 7 30	(530) 170 17 30				
			480 17 25	60 2 10·6	−0·16	0·03
4 times *with* the graduations	290 34 30 5 20	(650) 290 34 55		$\frac{53·8}{5}$ = 10·76.		5·78

$$\sqrt{\frac{5·78}{(5-1)5}} = \sqrt{0·29} = \pm 0·54.$$

$$\psi = 60° \; 2' \; 10·8'' \pm 0·5''.$$

B. TEST PRISM OF CROWN, No. 506

II. REFRACTION FOR LINE D

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation ×6 = 6γ.	Angle of Deviation = γ.	Error.	Error Squared.
Setting	304° 1′ 10″ 2 20	304° 1′ 45″				
3 times *with* the graduations	175 50 10 0 20	(535) 175 50 15	231° 48′ 30″	38° 38′ 5·0″	−9·3	86·49
3 times *with* the graduations	47 40 0 30	(407) 47 40 15	231 50 0	38 38 20·0	+5·7	32·49
3 times *with* the graduations	279 29 20 30 10	279 29 45	231 49 30	38 38 15·0	+0·7	0·49
3 times *with* the graduations	151 19 10 9 10	(511) 151 19 10	231 49 25	38 38 14·2	−0·1	0·01
3 times *with* the graduations	23 8 30 9 20	(383) 23 8 55	231 49 45	38 38 17·5	+3·2	10·24
				$\frac{71{\cdot}7}{5} = 14{\cdot}3$		129·72

$$\sqrt{\frac{129{\cdot}72}{(5-1)5}} = \sqrt{6{\cdot}49} = \pm 2{\cdot}55$$

$$_{\mathrm{D}}\gamma = 38^\circ\ 38'\ 14{\cdot}3'' \pm 2{\cdot}5''.$$

B. TEST PRISM OF CROWN, No. 506

III. REFRACTION FOR LINE F

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation ×6 = 6γ.	Angle of Deviation = γ.	Error.	Error Squared.
Setting	21° 34′ 30″ 5 20	21° 34′ 55″				
3 times *with* the graduations	256 39 10 40 0	256 39 35	235° 4′ 40″	39° 10′ 46·7″	−5·1	26·01
3 times *with* the graduations	131 45 10 5 20	(491) 131 45 15	235 5 40	39 10 56·7	+4·9	24·01
3 times *with* the graduations	6 50 10 1 0	(366) 6 50 35	235 5 20	39 10 53·3	+1·5	2·25
3 times *with* the graduations	241 55 40 6 30	241 56 5	235 5 30	39 10 55·0	+3·2	10·24
3 times *with* the graduations	117 0 40 1 0	(477) 117 0 50	235 4 45	39 10 47·5	−4·3	18·49
				$\frac{259\cdot2}{5} = 51\cdot8$		81·00

$$\sqrt{\frac{81\cdot00}{(5-1)5}} = \sqrt{4\cdot05} = \pm 2\cdot01$$

$$_{F}\gamma = 39^\circ\ 10'\ 51\cdot8'' \pm 2\cdot0''.$$

B. TEST PRISM OF CROWN, No. 506

IV. REFRACTION FOR LINE G_1*

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation $\times 6 = 6\gamma$.	Angle of Deviation $= \gamma$.	Error.	Error Squared.
Setting	117° 0′ 40″ 1 0	117° 0′ 50″				
3 times *with* the graduations	355 27 50 8 40	355 28 15	238° 27′ 25″	39° 44′ 34·2″	− 6·5	42·25
3 times *with* the graduations	233 55 50 6 30	(593) 233 56 10	238 27 55	39 44 39·2	− 1·5	2·25
3 times *with* the graduations	112 24 30 4 40	(472) 112 24 35	238 28 25	39 44 44·2	+ 3·5	12·25
3 times *with* the graduations	350 52 30 3 40	350 53 5	238 28 30	39 44 45·0	+ 4·3	18·49
3 times *with* the graduations	229 20 50 1 30	(589) 229 21 10	238 28 5	39 44 40·8	+ 0·1	0·01
				$\frac{203\cdot4}{5} = 40\cdot7$		75·25

$$\sqrt{\frac{75\cdot25}{(5-1)5}} = \sqrt{3\cdot76} = \pm 1\cdot94$$

$$_{G_1}\gamma = 39^\circ\ 44'\ 40\cdot7'' \pm 1\cdot9''.$$

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* This line does not coincide with the Fraunhofer G line.

B. TEST PRISM OF CROWN, No. 506

V. REFRACTION FOR LINE H

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation ×6 = 6γ.	Angle of Deviation = γ.	Error.	Error Squared.
Setting	289° 25′ 0″ 5 50	289° 25′ 25″				
			240° 22′ 40″	40° 3′ 46·7″	− 2·7	7·29
3 times *with* the graduations	169 48 0 8 10	(529) 169 48 5				
			240 23 5	40 3 50·8	+ 1·4	1·96
3 times *with* the graduations	50 10 50 1 30	(410) 50 11 10				
			240 21 55	40 3 39·2	− 10·2	104·04
3 times *with* the graduations	290 32 40 3 30	290 33 5				
			240 24 5	40 4 0·8	+ 11·4	129·96
3 times *with* the graduations	170 57 10 7 10	(530) 170 57 10				
			240 23 5	40 3 50·8	+ 1·4	1·96
3 times *with* the graduations	51 20 0 0 30	(411) 51 20 15				
			240 22 50	40 3 48·3	− 1·1	1·21
3 times *with* the graduations	291 42 30 3 40	291 43 5		$\frac{296\cdot6}{6}$	= 49·4	246·42

$$\sqrt{\frac{246\cdot42}{(6-1)6}} = \sqrt{8\cdot21} = \pm 2\cdot86$$

$$_{H}\gamma = 40^\circ\ 3'\ 49\cdot4'' \pm 2\cdot9''.$$

I. FOR FLINT GLASS, No. 318

	D.	F.	G_1.	H.
γ	46° 31′ 4·15″	47° 35′ 59·2″	48° 47′ 29·6″	49° 30′ 5·7″
ψ	59 56 22·4	59 56 22·4	59 56 22·4	59 56 22·4
$\gamma+\psi$	106 27 26·55	107 32 21·6	108 43 52·0	109 26 28·1
$\frac{\gamma+\psi}{2}$	53 13 43·3	53 46 10·8	54 21 56·0	54 43 14·05
$\frac{\psi}{2}$	29 58 11·2	29 58 11·2	29 58 11·2	29 58 11·2
$\log\sin\frac{\gamma+\psi}{2}$	9·903 6495	9·906 6839	9·909 9574	9·911 8737
$\log\sin\frac{\psi}{2}$	9·698 5730	9·698 5730	9·698 5730	9·698 5730
$\log n$	0·205 0765	0·208 1109	0·211 3844	0·213 3007
n	1·603 528	1·614 771	1·626 988	1·634 183

II. FOR CROWN GLASS, No. 506

	D.	F.	G_1.	H.
γ	38° 38′ 14·3″	39° 10′ 51·8″	39° 44′ 40·7″	40° 3′ 49·4″
ψ	60 2 10·8	60 2 10·8	60 2 10·8	60 2 10·8
$\gamma+\psi$	98 40 25·1	99 13 2·6	99 46 51·5	100 6 0·2
$\frac{\gamma+\psi}{2}$	49 20 12·55	49 36 31·3	49 53 25·75	50 3 0·1
$\frac{\psi}{2}$	30 1 5·4	30 1 5·4	30 1 5·4	30 1 5·4
$\log\sin\frac{\gamma+\psi}{2}$	9·879 9861	9·881 7478	9·883 5561	9·884 5719
$\log\sin\frac{\psi}{2}$	9·699 2084	9·699 2084	9·699 2084	9·699 2084
$\log n$	0·180 7777	0·182 5394	0·184 3477	0·185 3635
n	1·516 274	1·522 437	1·528 790	1·532 370

The refractive indexes thus determined for the two types of glass will now be used in the design of an achromatic prism to be made from these glasses.

Achromatic Prism and the Secondary Spectrum

When two prisms of the same glass and the same refracting angle are placed together so that the prism edges are oppositely situated, the whole then forming a piece of parallel glass, it will easily be seen that the two prisms refract and disperse the light by equal amounts but in opposite directions, so that an incident parallel beam of white light remains parallel and white on emergence. If for one of the prisms a more strongly refracting glass is used, the angle of this prism may be so chosen that light of a particular colour—for example, the D line—will be deviated by an equal but opposite amount to that produced by the first prism. If now the dispersion is greater in the same ratio as the refraction, the other coloured rays will also emerge parallel to the D ray; but if the dispersion does not vary proportionally to the refraction, then either the refraction of the two prisms may be made equal, in which case a portion of the dispersion remains over; or the dispersion for two colours may be made equal, in which case, however, the opposing refractions are no longer equal. The refraction of one of the prisms, namely, the prism having the weaker dispersive power, will predominate. In the case of the usual crown and flint glasses, the latter have the stronger refraction and the stronger dispersive power. The dispersion, however, is not stronger in the same proportion as the refraction, but is generally considerably more. If a crown-glass prism having a refracting angle of 60° is constructed, and it is desired to annul its dispersion, it is necessary to select a flint prism that will produce a spectrum of equal length but of opposite arrangement. The angle of this prism will be considerably smaller than that of the crown prism, and when combined they will produce a colourless image, which in this case will be deviated in the direction in which the crown prism refracts the light. Such a double prism of crown and flint glass is called *an achromatic prism*.

Suppose it is desired to construct, from the types of glass for which the refractive indexes have already been determined, a double prism that will be achromatic for the colours D and F. Suppose, further, to take a particular example, that the refracting angle of the first prism made of crown glass is found by measurement to be 60° 2′ 10·8″. It will be assumed also that the light of colour D is to pass through this prism at minimum deviation, and that the two prisms are to be cemented together.

This special example of an achromatic prism has been chosen as it will be made use of later.

The refractive indexes of the glasses employed are:—

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CROWN GLASS:

$${}_{D}n = 1{\cdot}516\,274 \qquad \log {}_{D}n = 0{\cdot}180\,7777.$$

$${}_{F}n = 1{\cdot}522\,437 \qquad \log {}_{F}n = 0{\cdot}182\,5394.$$

FLINT GLASS:

$${}_{D}n' = 1{\cdot}603\,528 \qquad \log {}_{D}n' = 0{\cdot}205\,0765.$$

$${}_{F}n' = 1{\cdot}614\,771 \qquad \log {}_{F}n' = 0{\cdot}208\,1109.$$

From these are obtained the following values:—

$$\frac{{}_{D}n}{{}_{D}n'} = 0{\cdot}945\,586 \qquad \log \frac{{}_{D}n}{{}_{D}n'} = 9{\cdot}975\,7012.$$

$$\frac{{}_{F}n}{{}_{F}n'} = 0{\cdot}942\,819 \qquad \log \frac{{}_{F}n}{{}_{F}n'} = 9{\cdot}974\,4285.$$

$$\Delta n = {}_{F}n - {}_{D}n = 0{\cdot}006\,163 \qquad \log \Delta n = 7{\cdot}789\,7922.$$

$$\Delta n' = {}_{F}n' - {}_{D}n' = 0{\cdot}011\,243 \qquad \log \Delta n' = 8{\cdot}050\,8822.$$

And $\dfrac{\Delta n}{\Delta n'} = 0{\cdot}548\,163 \qquad \log \dfrac{\Delta n}{\Delta n'} = 9{\cdot}738\,9100.$

In the first instance an approximate value of the flint prism angle should be calculated on the assumption that the prism angles are inversely proportional to the dispersions. Thus—

$$(2) \qquad \psi_3 = \psi_1 \frac{\Delta n}{\Delta n'}.$$

This equation gives the exact value of the ratio of the angles of the component prisms of an achromatic doublet, when the angles are small. The truth of this can easily be verified.

From formula (2)

$$\log \psi_1 = \log (60^\circ\, 2'\, 10{\cdot}8'') = \log 216\,131'' = 5{\cdot}334\,72.$$

$$\log \frac{\Delta n}{\Delta n'} = 9{\cdot}738\,91.$$

$$\log \psi_3 = 5{\cdot}073\,63.$$

$$\psi_3 = 118\,475'' = 32^\circ\, 54'\, 35''.$$

Commencing with this approximate value of the flint prism angle, rays of the two chosen colours should be reckoned through the double prism. When the normal drawn through the point of intersection of the incident ray with the first refracting surface lies between the ray and the refracting angle of the prism, as in fig. 19, ϕ_0 should be regarded as positive, in which case for the prism angle ψ_1 there should be chosen the upper sign of the double symbol placed before the angle ψ, and in the contrary position of the prism, the lower. (See Plate, p. 42.)

The change from ϕ_{2s} to ϕ'_{2s} follows the law of refraction, and therefore is obtained from the equation—

$$(3) \qquad \frac{\phi_{2s}}{\phi'_{2s}} = \frac{n_{2s-1}}{n_{2s+1}}.$$

The change from ϕ'_{2s} to ϕ_{2s+2} is in accordance with the formula

$$(4) \qquad \phi_{2s+2} = \phi'_{2s} \mp \psi_{2s+1}.$$

Finally, the deviations are determined from the equation

$$(5) \qquad \phi_{2s} - \phi'_{2s} = \delta_{2s}.$$

In the case of the crown prism, as already specified above, the light corresponding with the D line must suffer minimum deviation; that is, rays of the D line must pass symmetrically through the prism. Therefore

$$\phi'_0 = -\phi_2.$$

Also, since

$$(6) \qquad \psi_1 = \phi'_0 - \phi_2,$$

therefore

$$\psi_1 = 2\,\phi'_0, \text{ or } \phi'_0 = \frac{\psi_1}{2} = 30^\circ\, 1'\, 5{\cdot}4''.$$

It is now possible to determine the angle of incidence ϕ_0, thus:

$$\begin{aligned}\log\sin\phi'_0 &= 9{\cdot}699\,2084\\ \log {}_{\mathrm{D}}n &= 0{\cdot}180\,7777\\ \hline \log\sin\phi_0 &= 9{\cdot}879\,9861\\ \phi_0 &= 49^\circ\ 20'\ 12{\cdot}5''.\end{aligned}$$

With the aid of this value of the angle of incidence ϕ_0, there may be traced trigonometrically through the double prism the path of the light for the D and F colours, taking in the first instance the approximate value of the prism angle $\psi_3 = 32^\circ\ 54'\ 35''$. Thus there is obtained on p. 72 a total deviation for D of $14^\circ\ 41'\ 49{\cdot}8''$, and for F of $14^\circ\ 41'\ 57{\cdot}7''$.

In order that the deviations shall be the same, the angle ψ_3 of the flint prism must be made greater. By trial and error, the prism angle ψ_3 may be altered until finally the exact value is obtained at which the two rays of different colour are parallel to one another on emergence from the prism.

The doublet prism thus computed, and consisting of a crown prism of refracting angle $60^\circ\ 2'\ 10{\cdot}8''$ and of a flint prism of $33^\circ\ 6'\ 2{\cdot}5''$, causes the two rays of colours D and F to emerge parallel to each other and at an angle of $7^\circ\ 49'\ 15{\cdot}4''$ to the normal to the last prism surface, the total deviation being thus $14^\circ\ 34'\ 48{\cdot}8''$.

Not only does the prism produce no dispersion in the case of rays of colour D and F, but it would be achromatic for the other colours if the dispersion ratios for all the colours were the same; that is, if

$$(7)\qquad \frac{{}_{\mathrm{A}}n - {}_{\mathrm{B}}n}{{}_{\mathrm{A}}n' - {}_{\mathrm{B}}n'} = \frac{{}_{\mathrm{B}}n - {}_{\mathrm{C}}n}{{}_{\mathrm{B}}n' - {}_{\mathrm{C}}n'} = \ldots . \frac{{}_{\mathrm{G}}n - {}_{\mathrm{H}}n}{{}_{\mathrm{G}}n' - {}_{\mathrm{H}}n'}.$$

In general this is not so for such glasses as are available at the present time. Thus, for example, in the case of the two types of glasses under consideration,

$$\frac{{}_{\mathrm{D}}n - {}_{\mathrm{F}}n}{{}_{\mathrm{D}}n' - {}_{\mathrm{F}}n'} = 0{\cdot}548;\ \frac{{}_{\mathrm{F}}n - {}_{\mathrm{G}}n}{{}_{\mathrm{F}}n' - {}_{\mathrm{G}}n'} = 0{\cdot}520;\ \frac{{}_{\mathrm{G_1}}n - {}_{\mathrm{H}}n}{{}_{\mathrm{G_1}}n' - {}_{\mathrm{H}}n'} = 0{\cdot}498.$$

COMPUTATION OF AN ACHROMATIC PRISM

	D.	F.	D.	F.	D.	F.
ϕ_0	49° 20′ 12·5″	49° 20′ 12·5″				
log sin ϕ_0	9·879 9861	9·879 9861				
log (1 : n)	9·819 2223	9·817 4606				
log sin ϕ_0'	9·699 2084	9·697 4467				
ϕ_0'	30° 1′ 5·4″	29° 53′ 3·3″				
$\mp\psi_1$	− 60 2 10·8	− 60 2 10·8				
ϕ_2	− 30 1 5·4	− 30 9 7·5				
log sin ϕ_2	9·699 2084*n*	9·700 9605*n*				
log (n : n')	9·975 7012	9·974 4285				
log sin ϕ_2'	9·674 9096*n*	9·675 3890*n*				
ϕ_2'	− 28° 13′ 57·5″	− 28° 15′ 59·8″				
$\mp\psi_3$	32 54 35	32 54 35	33° 14′ 35″	33° 14′ 35″	33° 6′ 2·5″	33° 6′ 2·5″
ϕ_4	4 40 37·5	4 38 35·2	5 0 37·5	4 58 35·2	4 52 5	4 50 2·7
log sin ϕ_4	8·911 3705	8·908 2110	8·941 1974	8·938 2503	8·928 7103	8·925 6761
log n'	0·205 0765	0·208 1109	0·205 0765	0·208 1109	0·205 0765	0·208 1109
log sin ϕ_4'	9·116 4470	9·116 3219	9·146 2739	9·146 3612	9·133 7868	9·133 7870
ϕ_4'	7° 30′ 46·9″	7° 30′ 39·0″	8° 3′ 2·0″	8° 3′ 8·0″	7° 49′ 15·4	7° 49′ 15·4″
$\delta_0 = \phi_0 - \phi_0'$	+ 19 19 7·1	19 27 9·2			+ 19 19 7·1	+ 19 27 9·2
$\delta_2 = \phi_2 - \phi_2'$	− 1 47 7·9	− 1 53 7·7			− 1 47 7·9	− 1 53 7·7
$\delta_4 = \phi_4 - \phi_4'$	− 2 50 9·4	− 2 52 3·8	− 3 2 24·5	− 3 4 32·8	− 2 57 10·4	− 2 59 12·7
$\gamma_1 - \gamma_3 = \delta_0 + \delta_2 + \delta_4$	+ 14 41 49·8	+ 14 41 57·7	+ 14 29 34·7	+ 14 29 28·7	+ 14 34 48·8	+ 14 34 48·8

More recently types of glasses have been produced which give spectra that are nearly proportional between the C and F lines.

In the case of irrational spectra of types of glass to be used for the construction of achromatic prisms, achromatized for example for the colours D and F, it will be found that coloured rays lying between D and F, as well as those outside these lines, emerge at different angles. There is accordingly residual dispersion for the rays of colours other than D and F. Chromatic appearances of this kind are referred to as the *secondary spectrum.*

Determination of the Ratio of Dispersions of Two Types of Glasses

From the foregoing it will be seen that there are various values of $\frac{\Delta n}{\Delta n'}$ obtainable for two types of glasses. It is therefore necessary to find what value of $\frac{\Delta n}{\Delta n'}$ should be used in the computation in order that the best effect may be obtained for any given case, as, for example, a visual optical combination.

Fraunhofer has deduced the arithmetical mean from the individual values of the dispersion ratio of any one particular line and the next adjacent, taking into account the light intensity of the rays in question. If the light intensities for the colours A, B . . . H, J, are denoted by ${}_{A}K$, ${}_{B}K$ . . . ${}_{H}K$, ${}_{J}K$, and if, further, the total quantity of light between the lines is denoted by ${}_{A,B}K$, ${}_{B,C}K$ . . . ${}_{G,H}K$, ${}_{H,J}K$, then, according to Fraunhofer,

$$(8) \quad \frac{\Delta n}{\Delta n'} = \frac{\frac{{}_{A}n - {}_{B}n}{{}_{A}n' - {}_{B}n'}\,{}_{A,B}K + \frac{{}_{B}n - {}_{C}n}{{}_{B}n' - {}_{C}n'}\,{}_{B,C}K + \ldots}{{}_{A,B}K + {}_{B,C}K + \ldots}$$

$$\frac{+\frac{{}_{G}n - {}_{H}n}{{}_{G}n' - {}_{H}n'}\,{}_{G,H}K + \frac{{}_{H}n - {}_{J}n}{{}_{H}n' - {}_{J}n'}\,{}_{H,J}K}{+\,{}_{G,H}K + {}_{H,J}K}.$$

Assuming that the light intensities as observed by Fraunhofer, namely:

$$
{}_{A,B}K = 0{\cdot}000 \quad {}_{B,C}K = 0{\cdot}021 \quad {}_{C,D}K = 0{\cdot}299 \quad {}_{D,E}K = 1{\cdot}000
$$
$$
{}_{E,F}K = 0{\cdot}328 \quad {}_{F,G}K = 0{\cdot}185 \quad {}_{G,H}K = 0{\cdot}035 \quad {}_{H,J}K = 0{\cdot}000
$$

are applicable also to the particular types of glasses under consideration, namely, Crown No. 506, and Flint No. 318, the value of the quotient $\frac{\Delta n}{\Delta n'}$ may be calculated as follows:—

For Crown Glass.		For Flint Glass.
${}_{B}n = 1{\cdot}512\,67$	....	${}_{B}n' = 1{\cdot}597\,26$
${}_{C}n = 1{\cdot}513\,65$	....	${}_{C}n' = 1{\cdot}598\,93$
${}_{D}n = 1{\cdot}516\,27$	....	${}_{D}n' = 1{\cdot}603\,53$
${}_{E}n = 1{\cdot}519\,57$	....	${}_{E}n' = 1{\cdot}609\,45$
${}_{F}n = 1{\cdot}522\,44$	....	${}_{F}n' = 1{\cdot}614\,77$
${}_{G}n = 1{\cdot}527\,77$	....	${}_{G}n' = 1{\cdot}625\,04$
${}_{H}n = 1{\cdot}532\,37$	....	${}_{H}n' = 1{\cdot}634\,18$

Substituting these values in Equation (8),

$$
\frac{\Delta n}{\Delta n'} = \frac{\frac{0{\cdot}00098}{0{\cdot}00167}\,0{\cdot}021 + \frac{0{\cdot}00262}{0{\cdot}00460}\,0{\cdot}299 + \frac{0{\cdot}00330}{0{\cdot}00592}\,1{\cdot}000 +}{0{\cdot}021 + 0{\cdot}299 + 1{\cdot}000 +}
$$
$$
\frac{+\frac{0{\cdot}00287}{0{\cdot}00532}\,0{\cdot}328 + \frac{0{\cdot}00533}{0{\cdot}01027}\,0{\cdot}185 + \frac{0{\cdot}00460}{0{\cdot}00914}\,0{\cdot}035}{+\,0{\cdot}328 + 0{\cdot}185 + 0{\cdot}035.} = 0{\cdot}5517.
$$

According to C. A. Steinheil and L. Seidel,‡ a truer value of the quotient $\frac{\Delta n}{\Delta n'}$ is obtained from the formula

$$
(9)\quad \frac{\Delta n}{\Delta n'} = \frac{\frac{1}{2}\left(\frac{{}_{B}n - {}_{D}n}{{}_{B}n' - {}_{D}n'} + \frac{{}_{C}n - {}_{D}n}{{}_{C}n' - {}_{D}n'}\right){}_{B,C}K + \frac{1}{2}\left(\frac{{}_{C}n - {}_{D}n}{{}_{C}n' - {}_{D}n'} + \frac{{}_{E}n - {}_{D}n}{{}_{E}n' - {}_{D}n'}\right){}_{C,D}K +}{{}_{B,C}K + {}_{C,D}K +}
$$
$$
\frac{+\frac{1}{2}\left(\frac{{}_{C}n - {}_{D}n}{{}_{C}n' - {}_{D}n'} + \frac{{}_{E}n - {}_{D}n}{{}_{E}n' - {}_{D}n'}\right){}_{D,E}K + \frac{1}{2}\left(\frac{{}_{E}n - {}_{D}n}{{}_{E}n' - {}_{D}n'} + \frac{{}_{F}n - {}_{D}n}{{}_{F}n' - {}_{D}n'}\right){}_{E,F}K +}{+\,{}_{D,E}K + {}_{E,F}K +}
$$
$$
\frac{+\frac{1}{2}\left(\frac{{}_{F}n - {}_{D}n}{{}_{F}n' - {}_{D}n'} + \frac{{}_{G}n - {}_{D}n}{{}_{G}n' - {}_{D}n'}\right){}_{F,G}K + \frac{1}{2}\left(\frac{{}_{G}n - {}_{D}n}{{}_{G}n' - {}_{D}n'} + \frac{{}_{H}n - {}_{D}n}{{}_{H}n' - {}_{D}n'}\right){}_{G,H}K}{+\,{}_{F,G}K + {}_{G,H}K}.
$$

‡ See Appendix I.

Substituting in this formula (9) the above values of the refractive indexes and light intensities, there is obtained the result

$$\frac{\Delta n}{\Delta n'} = 0{\cdot}5590.$$

The most suitable value of $\frac{\Delta n}{\Delta n'}$ for a visual optical combination can be more easily found by the following simple observations. A cemented doublet prism achromatized for the lines D and F should be calculated for the two types of glasses, as already described on p. 68, and such a prism of approximately this calculated value should be constructed.

If, in the prism as constructed, the angle ψ_3 is not exactly realized, the actual value of the angle ψ_3 obtained should be used in the further computations.

An achromatic doublet prism has already been computed for the two types of glasses, Crown No. 506 and Flint No. 318, p. 72. It was found that for a crown prism of angle $\psi_1 = 60^\circ\ 2'\ 10{\cdot}8''$, a flint prism of angle $\psi_3 = 33^\circ\ 6'\ 2{\cdot}5''$ was required, in order to obtain achromatism of the colours D and F for the combined prism. A flint prism of approximately the calculated angle should be constructed, and the angle measured exactly on the goniometer. The observations thus obtained are tabulated on p. 77.

The flint-glass prism having a refracting angle of $33^\circ\ 35'\ 47{\cdot}3''$ should then be joined with castor oil to the crown-glass prism of refracting angle $60^\circ\ 2'\ 10{\cdot}8''$ previously made.

This double prism will produce an achromatic image of the collimator slit only when it is so placed upon the goniometer table that the angle of incidence of the parallel beam from the collimator at the first prism surface is the angle specified by the computation, namely, in this particular example, $\phi_0 = 49^\circ\ 20'\ 12{\cdot}5''$.

Under these conditions the crown prism is set at minimum deviation, but the compound prism is far from its position of minimum deviation. In this particular example,

$$\phi_0 = 49^\circ\ 20'\ 12{\cdot}5''$$

whereas $$\phi'_4 = 7^\circ\ 49'\ 15{\cdot}4''.$$

Accordingly the position of the colours of the slit image will alter appreciably if the angle of incidence is altered, and, further, the alteration will be in different directions according as the incidence angle is increased or diminished. Not only will the colour position change, but at the same time the deviation which the rays suffer will alter. This provides a means of finding the quotient $\frac{\Delta n}{\Delta n'}$ which will give the best optical total effect when using these two types of glasses. For this purpose, while viewing the slit through the observing telescope, the prism should be turned in order to find the position in which the slit image appears most sharply defined and achromatic. When in this position the deviation of the rays through the prism should be determined by reading the scale, and then, after removing the prism, by directing the telescope, together with the graduated circle, directly upon the collimator slit and again reading the scale. The difference of the two readings is the deviation for which the colour effect is best. In general it is more convenient to determine the deviation first from the one side and then from the other by suitably rotating the prism, and thus to observe the angle of deviation by repeated measurement in the manner already described for the determination of refractive indexes.

In a particular case the results of measurement for the determination of the angle of deviation were as indicated on p. 78.

SECOND TEST PRISM OF FLINT GLASS, No. 318

PRISM ANGLE

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Prism Angle ×6 = 6ψ.	Prism Angle = ψ.	Error.	Error Squared.
Setting	121° 14′ 0″ 4 10	121° 14′ 5″				
			201° 34′ 25″	33° 35′ 44·2″	−3·14	9·86
3 times *with* the graduations	322 48 0 9 0	322 48 30				
			201 34 50	33 35 48·3	+0·96	0·92
3 times *with* the graduations	164 23 30 3 10	(524) 164 23 20				
			201 35 0	33 35 50·0	+2·66	7·08
3 times *with* the graduations	5 58 0 8 40	(365) 5 58 20				
			201 34 55	33 35 49·2	+1·86	3·46
3 times *with* the graduations	207 33 10 3 20	207 33 15				
			201 34 30	33 35 45·0	−2·34	5·48
3 times *with* the graduations	49 7 30 8 0	(409) 49 7 45		$\frac{236·7}{5} = 47·34$		26·80

$$\sqrt{\frac{26·80}{(5-1)5}} = \sqrt{1·340} = \pm 1·16$$

$$\psi = 33^\circ\ 35'\ 47·3'' \pm 1·2''.$$

DOUBLE PRISM OF CROWN No. 506 AND FLINT No. 318

DEVIATION

	Readings on Verniers II and IV.	Corrected Reading on Vernier II.	Angle of Deviation ×6 = 6γ.	Angle of Deviation = γ.	Error.	Error Squared.
Setting	37° 35′ 10″ 5 30	37° 35′ 20″				
			81° 52′ 55″	13° 38′ 49·2″	+ 111·42	12414
3 times *with* the graduations	119 28 20 8 10	119 28 15				
			81 40 55	13 36 49·2	− 8·58	74
3 times *with* the graduations	201 9 10 9 10	201 9 10				
			81 52 15	13 38 42·5	+ 104·72	10966
3 times *with* the graduations	283 1 0 1 50	283 1 25				
			82 1 10	13 40 11·7	+ 193·92	37605
3 times *with* the graduations	5 2 20 2 50	(365) 5 2 35				
			81 48 55	13 38 9·2	+ 71·42	5101
3 times *with* the graduations	86 51 30 1 30	86 51 30				
			81 46 50	13 37 48·3	+ 50·52	2552
3 times *with* the graduations	168 38 20 8 20	168 38 20				
			81 42 40	13 37 6·7	+ 8·92	80
3 times *with* the graduations	250 20 40 1 20	250 21 0				
			81 2 15	13 30 22·5	− 395·28	156246
3 times *with* the graduations	331 22 40 3 50	331 23 15				
			81 58 15	13 39 42·5	+ 164·72	27133
3 times *with* the graduations	53 21 20 1 40	(413) 53 21 30				
			80 56 15	13 29 22·5	− 455·28	207280
3 times *with* the graduations	134 17 50 7 40	134 17 45				
			81 44 5	13 37 20·8	+ 23·02	530
3 times *with* the graduations	216 1 40 2 0	216 1 50				
			81 54 50	13 39 8·3	+ 130·52	17035
3 times *with* the graduations	297 56 10 7 10	297 56 40				
				$\frac{443 \quad 33{\cdot}4}{12} = 36' \ 57{\cdot}78''$		477016

$$\sqrt{\frac{477016}{(12-1)12}} = \sqrt{3613} = \pm 60{\cdot}1$$

$$\gamma = 13^\circ \ 36' \ 57{\cdot}8'' \pm 60{\cdot}1''.$$

It will be seen that no great accuracy has been attained in the setting of the prism for the best achromatic position, since the average error of the results is about thirty times as great as that obtained in the determination of the angle of the prism. The accuracy of the computation of $\frac{\Delta n}{\Delta n'}$ is, however, not affected in a high degree.

There should now be deduced by calculation, first, the angle of incidence corresponding with the measured deviation, and then, after having chosen a suitable value of the refraction for the crown glass, the correspondingly greater value n' of the more strongly refracted ray for the flint glass.

There are known the values of $_Dn$ for the crown glass, $_Dn'$ for the flint glass, the angle ψ_1 of the crown prism, and the angle ψ_3 of the flint prism. In commencing the calculation, the incidence angle at the first face is taken from the approximate calculation, being in this example 49° 20′ 12·5″. Then the path of the ray of colour D is traced through the three refractions which it suffers at the surfaces of the prism, and from the computation there is derived the total deviation, which in this particular case is 14° 16′ 31·2″. As this value is too great, in order to reduce the deviation the combined prism must be brought nearer to minimum deviation; that is, the angle of incidence must be given a smaller value. Using in the computation this new value for the angle of incidence, another value of the angle of deviation is obtained.

These figures for the angles of incidence and deviation, together with those previously obtained, make it possible to determine the effect upon the angle of deviation of a change in the angle of incidence, so that it is possible, by interpolation, to obtain a value of the angle of incidence very near to the correct one. The latter can be obtained exactly by repetition of the process.

In the example chosen the calculation should be carried out as follows:—

ϕ_0 log sin ϕ_0 log (1 : n)	49° 20′ 12·5″ 9·879 99 9·819 22	46° 36′ 30″ 9·861 34 9·819 22	46° 36′ 35″ 9·861 35 9·819 22
log sin ϕ_0' ϕ_0' $\pm\psi_1$	9·699 21 30° 1′ 5·4″ −60 2 10·8	9·680 56 28° 38′ 11·2″ −60 2 10·8	9·680 57 28° 38′ 13·8″ −60 2 10·8
ϕ_2 log sin ϕ_2 log (n : n')	−30 1 5·4 9·699 21n 9·975 70	−31 23 59·6 9·716 84n 9·975 70	−31 23 57·0 9·716 84n 9·975 70
log sin ϕ_2' ϕ_2' $\mp\psi_3$	9·674 91n −28° 13′ 57·5″ 33 35 47·3	9·692 55n −29° 30′ 55·6″ 33 35 47·3	9·692 54n −29° 30′ 53·2″ 33 35 47·3
ϕ_4 log sin ϕ_4 log n'	5 21 49·8 8·970 68 0·205 08	4 4 51·7 8·852 28 0·205 08	4 4 54·1 8·852 35 0·205 08
log sin ϕ_4' ϕ_4' δ_0 δ_2 δ_4	9·175 75 8° 37′ 17·8″ +19 19 7·1 − 1 47 7·9 − 3 15 28·0	9·057 36 6° 33′ 10·1″ +17 58 18·8 − 1 53 4·0 − 2 28 18·4	9·057 43 6° 33′ 13·9″ +17 58 21·2 − 1 53 3·8 − 2 28 19·8
$\gamma_{1,3}$	+14 16 31·2	+13 36 56·4	+13 36 57·6

Having found, as a result of the foregoing calculation, the value of the angle of incidence, namely, $\phi_0 = 46° 36′ 35″$, in this particular example, there should be chosen a value of n for the crown glass, the most suitable value being one corresponding with a colour beyond D in the direction of F, to ensure that the influence of the colour will be truly expressed even with few decimal places. In this particular case, for example, there will be chosen a refractive index of 1·527 77 for the crown glass, which will be denoted by ${}_{V}n$ where the suffix V represents violet rays.

If the difference

$$ {}_{V}n - {}_{D}n = 1{\cdot}527\,77 - 1{\cdot}516\,27 = 0{\cdot}011\,50 $$

be multiplied by the quotient

$$ \frac{{}_{F}n' - {}_{D}n'}{{}_{F}n - {}_{D}n} = \frac{1{\cdot}614\,77 - 1{\cdot}603\,53}{1{\cdot}522\,44 - 1{\cdot}516\,27} = \frac{0{\cdot}011\,24}{0{\cdot}006\,17}, $$

there is obtained an approximate value of

$$_{\mathrm{Y}}n' - {_{\mathrm{D}}n'} = 0{\cdot}020\,95,$$

and, since $_{\mathrm{D}}n' = 1{\cdot}603\,53$, it follows that the approximate value of $_{\mathrm{Y}}n'$ is 1·624 48.

Using these values and that of the first angle of incidence, which was found to be $\phi_0 = 46^\circ\ 36'\ 35''$, the double prism should again be computed, and it should be noted whether the emergent rays are parallel to the yellow ray. If this is not the case, the value of $_{\mathrm{Y}}n'$ should be varied until the emergent ray found by computation is parallel to the emergent D ray; that is, in this particular example, until the angle of deviation $\gamma_{1,3} = 13^\circ\ 36'\ 57{\cdot}8''$.

The computation is carried out as follows:—

ϕ_0	46° 36′ 35″		
log sin ϕ_0	9·861 35		
log (1 : n)	9·815 94		
log sin ϕ_0'	9·677 29		
ϕ_0'	28 24′ 7·2″		
$\pm\psi_1$	−60 2 10·8		
ϕ_2	−31 38 3·6		
log sin ϕ_2	9·719 74n	9·719 74n	9·719 74n
log (n : n')	9·973 34	9·973 50	9·973 48
log sin ϕ_2'	9·693 08n	9·693 24n	9·693 23n
ϕ_2'	−29° 33′ 29·7″	−29° 34′ 2·8″	−29° 33′ 58·5″
$\mp\psi_3$	33 35 47·3	33 35 47·3	33 35 47·3
ϕ_4	4 2 17·6	4 1 44·5	4 1 48·8
log sin ϕ_4	8·847 71	8·846 72	8·846 85
log n'	0·210 71	0·210 56	0·210 57
log sin ϕ_4'	9·058 42	9·057 28	9·057 42
ϕ_4'	6° 34′ 8·3″	6° 33′ 5·8″	6° 33′ 13·7″
δ_0	+18 12 27·8	+18 12 27·8	+18 12 27·8
δ_2	− 2 4 33·9	− 2 4 0·8	− 2 4 5·7
δ_4	− 2 31 50·7	− 2 31 21·3	− 2 31 24·3
$\gamma_{1,3}$	+13 36 3·2	+13 37 5·7	+13 36 57·8

The equation

$$(10)\qquad \frac{\Delta n}{\Delta n'} = \frac{{}_{\mathrm{V}}n - {}_{\mathrm{D}}n}{{}_{\mathrm{V}}n' - {}_{\mathrm{D}}n'} = \frac{0{\cdot}011\,50}{0{\cdot}020\,43} = 0{\cdot}5628$$

gives the value of $\frac{\Delta n}{\Delta n'}$, which corresponds with the best total effect.

Naturally the determination of the value $\frac{\Delta n}{\Delta n'}$, cannot be made by colour-blind persons, and red-blind persons, say, will obtain quite different results. Persons having eyes of this description require another arrangement of the colours in optical instruments, which will appear to them good. For example, red-blind people should neglect the red rays.

There is one principal difference between instruments to be used by the eyes and those for photographic purposes, namely, that in the former there have to be considered only those rays which act upon the eye and their appropriate intensity, and in the latter the more intense chemically active rays. Whereas in the case of the optically active rays the maximum intensity lies in the yellow portion of the spectrum, the chemically active rays mostly reach a maximum in the neighbourhood of the G line. Accordingly in photographic apparatus the violet rays must chiefly be considered, and in the majority of instances for the production of sharply-defined images the most intense optical rays may be left quite out of consideration. When this is the case it is sufficient to use the G and H lines in the computation. Cases arise, however, in which it is better to select those rays that have both maximum optical and chemical intensity, as, for example, in terrestrial photography at very various distances, where it is desirable to adjust the optical focus with special care, or when using ortho-chromatic plates, which are active to yellow light.

CHAPTER II

Achromatic Doublet Objectives

Conditions for the Computation of Doublet Objectives

In Chapter IV of the first volume it has been shown what results can be obtained from simple lenses and what are the residual errors in the images produced by them. In that chapter there were computed various forms of lenses in which the distribution of the refraction over the two faces was varied by altering the radii of the surfaces, the lenses being of the same glass and having the same aperture and true focal length and also the same thickness, in accordance with the aperture. It was also shown wherein the images produced by these lenses differed from one another. It appeared that the definition of an image point *on the axis* was very considerably affected by two errors, namely, chromatic and spherical aberrations, and that, although these errors could be reduced to a minimum, they could not be entirely removed. On account of the conditions specified all the elements of the lens were fixed, with the exception of the two radii, and accordingly for a definite value of the first radius only *one* condition, namely the obtaining of the true focal length, could be fulfilled by means of the second radius.

In the addition of a second lens the type of glass will be so chosen as to make the achromatization of the lens combination possible. Suppose, further, that the combinations to be computed are to have the same aperture and the same true focal length. The thickness of the positive lens should correspond with the desired aperture, and should be the same in all cases. The thickness of the negative lens, which must be such that its strength will be sufficient, will be assumed to be the same in all cases. Lastly, the separation of the two lenses will be made as small as possible.

With such a combination of two lenses, two further conditions can now be satisfied, corresponding with the two additional radii; that is, in addition to obtaining the true focal length, the chromatic and spherical aberrations may be removed.

In order to achromatize a lens system, while obtaining the desired positive true focal length; it is necessary that the dispersions of the glasses employed should vary in a different manner from the refractive indexes. For, if the refractions and dispersions of the glasses varied proportionally, it would be necessary, in order to produce a chromatic aberration equal to that of the positive lens, that the negative lens should have an equal but opposite true focal length. The lens combination would then be equivalent to a plane parallel glass. Its total deviation would be zero and the combined focal length infinite. By reason of the unproportionality of the dispersions in relation to the refractions it is possible to leave uncompensated a considerable portion of the refraction produced by the positive lens, when the chromatic aberration is eliminated by means of a negative lens made of glass of stronger dispersive power. Further, it will be clear that the greater the difference between the dispersions of the two glasses the weaker must be the refractions of the lenses, and the smaller this difference the stronger must be the refractions, in order to obtain an achromatic combination of a specified true focal length.

As is well known, the individual portions of the spectra of various types of glasses are not strictly proportional to one another. So-called achromatic doublet combinations made from such glasses are accordingly not perfectly achromatic. Indeed, the condition of achromatism can only be completely fulfilled for two chosen colours. The choice of these colours is determined by the purpose which the achromatic objective has to fulfil, as indicated on p. 82.

In the further consideration of the subject there will be chosen two types of glasses, corresponding with those in most general use. From these glasses there will be computed an achromatic objective comprising a positive lens of crown glass and a negative lens of flint, such that the combination will have a prescribed positive focal length, and that the individual lenses will have chromatic and spherical aberrations of equal but opposite amounts.

In all cases it is possible to obtain a desired true focal length and achromatism, since corresponding with any crown lens there can be found a flint lens, and with any flint lens a particular form of crown lens, that will satisfy these conditions. It is otherwise, however, in the case of spherical aberration. As is well known, this error is due to the fact that the refraction is not proportional to the angle of incidence, but to the sine of this angle, and accordingly the refraction at spherical surfaces is not proportional to the incidence heights, but increases more quickly than the latter. From this it follows that the spherical aberration of a lens must be the greater the more unequally the refraction is divided over the two surfaces of the lens, and that it reaches a minimum when the deviations at the two surfaces are equally great and in the same sense. Further, when it is considered that the deviation of the positive lens must always be greater than that of the negative in the case of a positive achromatic system, it follows that the minimum spherical aberration for the positive lens producing equal refractions at the two surfaces has a greater value than the minimum spherical aberration of the flint lens having equal refracting surfaces. The spherical aberration can, however, only be removed when the errors in the two lenses are equal and opposite. Accordingly, for any crown lens there can be found a flint lens that will eliminate the spherical aberration, since the error of the simple equally-refracting flint lens is smaller, and therefore by an unequal distribution of the refractions the aberration can be increased until it equals that of the crown lens. On the other hand, corresponding with a flint lens having equally-refracting surfaces, there cannot be found a corresponding crown lens which has an equally small but opposite error. In systems which have to be free from spherical aberration, only those flint lenses can be used which, in consequence of the unequal distribution of the refraction over the two surfaces, give rise to spherical aberration that is as great as, or greater than, that of the crown lens having equally refracting surfaces.

The various possible lens combinations can be divided into two groups according as the crown or the flint lens is in front.

The values of the deviations of the two lenses, which when combined form an achromatic objective, are determined by the types of glasses chosen. When these values are known, the first radius of the front lens can be chosen at will; the second radius must then be such that the lens is of the desired power. If the front lens is of crown glass, there may be calculated for each form, whose radii give the desired aperture without exceeding the prescribed thickness, two flint lenses which remove the spherical and chromatic aberrations, the stronger dispersion occurring in the one case at the first surface, and in the other case at the second. When, however, the front lens is of flint glass, all those forms no longer arise in which the spherical aberration is smaller than that of the equally refracting crown lens. The two examples of flint lenses in which the error is the same as that of the equally refracting crown lens permit of the use of only one crown lens; that is, the lens having equal refractions at the two surfaces. On the other hand, all flint lenses of greater spherical aberration can be combined with either of two crown lenses to form an objective which, while having the prescribed focal length, will be free from chromatic and spherical aberration.

Computation of Doublet Objectives

As an example of the computation of doublet achromatic objectives which will also be free from spherical aberration, there will be chosen an objective having the crown lens in front and made of the following types of glass, namely:—

CROWN GLASS: $_{D}n = 1{\cdot}518\,06$
$_{V}n = 1{\cdot}533\,56$
FLINT GLASS: $_{D}n' = 1{\cdot}613\,58$
$_{V}n' = 1{\cdot}642\,52$

The dimensions of the objective are to be as follows:—

Semi-aperture	30 mm.
True focal length	720 ,,
Axial thickness of the positive lens	8 ,,
Axial thickness of the negative lens	5 ,,
Separation of the lenses	0·01 ,,

In those forms of objective which enclose a positive lens of air between the glass components, the separation must be chosen to correspond with the prescribed aperture in order to prevent the occurrence of impossible cases in the computation.

For the calculation of the radii of the objective there will be chosen two incident rays parallel to the axis: one, the paraxial ray, lying near the axis; and the other, the rim ray, incident near the periphery of the lens. As these white incident rays suffer dispersion at the first refraction, there will be traced thereafter two component coloured rays, one corresponding with the D line, and one corresponding with the more strongly refracted violet light (see p. 82). For the D ray the true focal length is computed and the spherical aberration removed. The choice of the first radius of the lens system determines the construction of the combination, the second the achromatism, the correct choice of the third eliminates the spherical aberration, while the fourth must be so chosen that the true focal length is of the prescribed value.

At first, until the chromatic aberration is removed, it is sufficient to compute only the axial rays and to determine the chromatic aberration given by the approximately correct values. In order to remove the chromatic aberration, the second radius should be altered in such a way that the deviation of each lens is increased by an equal amount when the chromatic aberration is uncorrected, and decreased when over-corrected.

After the chromatic aberration has been removed in this way, the rim ray for the colour D should be computed and thereby the spherical aberration determined. The latter can then be removed by altering the distribution of the refraction over the two last surfaces more unequally when the error is uncorrected, and more equally when over-corrected. If, as a result of this, the chromatic aberration is somewhat altered, the second radius must again be returned to. The difference between the second and third radii must be kept the same as in the case when the spherical aberration was zero, and the computation must again be carried through, the fourth radius being used to obtain the desired focal length.

If the construction of the objective requires to be altered, it is necessary to modify the first radius; to correct the chromatic aberration, the second radius should be altered; to correct the spherical aberration, the third should be changed; and in all cases the true focal length is obtained by a variation of the last radius.

Having briefly described in the foregoing the scheme of the calculations, there will now be carried out the actual computation of the example specified above, of which the given data are:

$$h_0 = 30 \text{ mm.}; \; f_{0,6} = 720 \text{ mm.}; \; d_1 = 8{\cdot}0 \text{ mm.};$$
$$d_3 = 0{\cdot}01 \text{ mm.}; \; d_5 = 5{\cdot}0 \text{ mm.}$$

For an object at infinity

$$_{\mathrm{D}}n = 1{\cdot}518\,06; \quad \log {}_{\mathrm{D}}n = 0{\cdot}181\,289$$
$$_{\mathrm{V}}n = 1{\cdot}533\,56; \quad \log {}_{\mathrm{V}}n = 0{\cdot}185\,701$$
$$_{\mathrm{D}}n' = 1{\cdot}613\,58; \quad \log {}_{\mathrm{D}}n' = 0{\cdot}207\,791$$
$$_{\mathrm{V}}n' = 1{\cdot}642\,52; \quad \log {}_{\mathrm{V}}n' = 0{\cdot}215\,511$$

From this it follows that

$$_{\mathrm{V}}n - {}_{\mathrm{D}}n = \Delta n = 0{\cdot}015\,50; \qquad \log \Delta n = 8{\cdot}190\,332$$
$$_{\mathrm{V}}n' - {}_{\mathrm{D}}n' = \Delta n' = 0{\cdot}028\,94; \qquad \log \Delta n' = 8{\cdot}461\,499$$
$$\log \frac{\Delta n'}{\Delta n} = 0{\cdot}271\,167; \qquad \frac{\Delta n'}{\Delta n} = 1{\cdot}867\,11$$
$$_{\mathrm{D}}n - 1 = 0{\cdot}518\,06; \qquad \log ({}_{\mathrm{D}}n - 1) = 9{\cdot}714\,380$$
$$_{\mathrm{D}}n' - 1 = 0{\cdot}613\,58; \qquad \log ({}_{\mathrm{D}}n' - 1) = 9{\cdot}787\,871$$

As axial ray there will be chosen that one which, after passing through the lens combination, is inclined to the axis at an angle of 1″, so that for this ray, $\beta_{0,6} = 1''$. Since $\beta_{0,6} = \frac{h_0}{f_{0,6}}$, the incidence height for this paraxial ray must be taken as $h_0 = 720$. (Cf. p. 62, Vol. I.) Accordingly

$$\beta_{0,6} = \frac{h_0}{f_{0,6}} = \frac{720}{720} = 1.$$

The total deviation $\beta_{0,6}$ of the lens combination comprises the deviations $\beta_{0,2}$ and $\beta_{4,6}$ of the individual lenses; therefore

$$\beta_{0,2} + \beta_{4,6} = \beta_{0,6} = 1.$$

In order that the combination may form an achromatic image for axial rays, the deviations of the individual lenses must satisfy the approximately correct conditions (see p. 69), namely:

$$(11)\qquad \beta_{0,2} = \frac{1}{1 - \dfrac{({}_{D}n' - 1)\Delta n}{({}_{D}n - 1)\Delta n'}} \text{ and } \beta_{4,6} = -\frac{1}{\dfrac{({}_{D}n - 1)\Delta n'}{({}_{D}n' - 1)\Delta n} - 1}.$$

Inserting the above-mentioned values in these equations, it follows that:

$$\beta_{0,2} = +2{\cdot}734\,803''. \qquad \log \beta_{0,2} = 0{\cdot}436\,926.$$
$$\beta_{4,6} = -1{\cdot}734\,803''. \qquad \log \beta_{4,6} = 0{\cdot}239\,250n.$$

The first radius of the combination may be chosen at will. In this example let $r_0 = 420{\cdot}00$ mm. By means of the formula applicable to an axial ray (see Vol. I, p. 63), namely:

$$(12)\qquad r_0 = \frac{h_0({}_{D}n - 1)}{\delta_0\,{}_{D}n},$$

there may be determined the values of δ_0 and δ_2, since $\delta_0 + \delta_2 = \beta_{0,2}$, the value $h_0 = 720$ mm. being used as previously mentioned.

Thus:

$$\begin{array}{rcr}
\log h_0 & = & 2{\cdot}857\,333 \\
\log({}_{D}n - 1) & = & 9{\cdot}714\,380 \\
\log(1 : {}_{D}n) & = & 9{\cdot}818\,711 \\
\log(1 : r_0) & = & 7{\cdot}376\,751 \\ \hline
\log \delta_0 & = & 9{\cdot}767\,175 \\
-\delta_0 & = & -0{\cdot}585\,026 \\
\beta_{0,2} & = & 2{\cdot}734\,803 \\ \hline
\delta_2 & = & 2{\cdot}149\,777 \\
\log \delta_2 & = & 0{\cdot}332\,393
\end{array}$$

The radius r_2 is obtained from the formula:

$$(13)\qquad r_2 = \frac{(h_0 - d_1\,\delta_0)({}_{D}n' - 1)}{-\delta_2 + \delta_0({}_{D}n - 1)}.$$

(See Vol. I, p. 63.)

Thus:

$$
\begin{aligned}
\log d_1 &= 0{\cdot}903\,090 \\
\log \delta_0 &= 9{\cdot}767\,175 \\
\log (d_1\delta_0) &= 0{\cdot}670\,265 \\
-d_1\delta_0 &= -4{\cdot}680\,206 \\
h_0 &= 720 \\
h_0 - d_1\delta_0 &= 715{\cdot}319\,794 \\
\log (h_0 - d_1\delta_0) &= 2{\cdot}854\,500 \\
\log ({}_{D}n - 1) &= 9{\cdot}714\,380 \\
\log (h_0 - d_1\delta_0)({}_{D}n - 1) &= 2{\cdot}568\,880 \\
\log \delta_0 &= 9{\cdot}767\,175 \\
\log ({}_{D}n - 1) &= 9{\cdot}714\,380 \\
\log \delta_0 ({}_{D}n - 1) &= 9{\cdot}481\,555 \\
\delta_0 ({}_{D}n - 1) &= 0{\cdot}303\,078 \\
-\delta_2 &= -2{\cdot}149\,777 \\
-\delta_2 + \delta_0 ({}_{D}n - 1) &= -1{\cdot}846\,699 \\
\log [-\delta_2 + \delta_0 ({}_{D}n - 1)] &= 0{\cdot}266\,396n \\
\log (h_0 - d_1\delta_0)({}_{D}n - 1) &= 2{\cdot}568\,880 \\
\log \{1 : [-\delta_2 + \delta_0 ({}_{D}n - 1)]\} &= 9{\cdot}733\,604n \\
\log r_2 &= 2{\cdot}302\,484n \\
r_2 &= -\ 200{\cdot}671
\end{aligned}
$$

The radius r_4 will be chosen somewhat longer than the radius r_2, in order that the lens surfaces, separated at the centre by the small distance of 0·01 mm., will be in contact around the edge.* Thus, for example, the radius r_4 will be made 5 mm. greater than r_2, that is,

$$r_4 = 200{\cdot}671 \text{ mm.} + 5 \text{ mm.} = 205{\cdot}671 \text{ mm.}$$

Working on the assumptions previously made, the paraxial ray can be trigonometrically traced through the lens combination until it emerges from the third surface. It is then possible to ascertain the hitherto unknown values of ϕ_6 and r_6, which must be so chosen as to give the true focal length. Finally, the trigonometrical computation should be carried out simultaneously for both the axial rays of colour D and of violet, in order to determine the residual chromatic aberration corresponding with the chosen data of the combination. In this calculation six-figure logarithms will be used, since with five-figure logarithms the chromatic aberration could not be calculated with sufficient accuracy.

---

* This statement as well as the similar one on p. 96 appears to be wrong. The radius r_4 should be made shorter than r_2, not longer, to ensure contact around the edge instead of the centre. As designed it would be necessary to separate the parts definitely by a suitable distance ring sufficient to prevent contact at the centre.—Ed.

The calculation is as follows:—

	Yellow.	Violet.
$\log h_0 =$	2·857 333	2·857 333
$\log (1 : r_0) =$	7·376 751	7·376 751
$\log \phi_0 =$	0·234 084	0·234 084
$\log (1 : n) =$	9·818 711	9·814 299
$\log \phi_0' =$	0·052 795	0·048 383
$\phi_0 =$	1·714 289	1·714 289
$-\phi_0' =$	−1·129 263	−1·117 849
$\delta_0 =$	0·585 026	0·596 440
$\log \delta_0 =$	9·767 175	9·775 567
$\log (1 : \delta_0) =$	0·232 825	0·224 433
$\log \phi_0' =$	0·052 795	0·048 383
$\log r_0 =$	2·623 249	2·623 249
$\log (a_0' - r_0) =$	2·908 869	2·896 065
$a_0' - r_0 =$	810·717	787·164
$r_0 - r_2 - d_1 =$	612·671	612·671
$a_0' - r_2 - d_1 =$	1 423·388	1 399·835
$\log (a_0' - r_2 - d_1) =$	3·153 323	3·146 077
$\log \delta_0 =$	9·767 175	9·775 567
$\log (1 : r_2) =$	7·697 516n	7·697 516n
$\log \phi_2 =$	0·618 014n	0·619 160n
$\log n =$	0·181 289	0·185 701
$\log \phi_2' =$	0·799 303n	0·804 861n
$\phi_2 =$	−4·149 674	−4·160 638
$-\phi_2' =$	6·299 455	6·380 593
$\delta_2 =$	2·149 781	2·219 955
$\delta_0 =$	0·585 026	0·596 440
$\beta_{0,2} =$	2·734 807	2·816 395

	Yellow.	Violet.
$\log \beta_{0,2} =$	0·436 927	0·449 694
$\log (1 : \beta_{0,2}) =$	9·563 073	9·550 306
$\log \phi_2' =$	0·799 303 *n*	0·804 861 *n*
$\log r_2 =$	2·302 484 *n*	2·302 484 *n*
$\log (a_2' - r_2) =$	2·664 860	2·657 651
$a_2' - r_2 =$	462·232 0	454·622 6
$r_2 - r_4 - d_3 =$	4·99	4·99
$a_2' - r_4 - d_3 =$	467·222 0	459·612 6
$\log (a_2' - r_4 - d_3) =$	2·669 523	2·662 392
$\log \beta_{0,2} =$	0·436 927	0·449 694
$\log (1 : r_4) =$	7·686 828 *n*	7·686 828 *n*
$\log \phi_4 =$	0·793 278 *n*	0·798 914 *n*
$\log (1 : n') =$	9·792 209	9·784 489
$\log \phi_4' =$	0·585 487 *n*	0·583 403 *n*
$\phi_4 =$	−6·212 666	−6·293 816
$-\phi_4' =$	3·850 233	3·831 802
$\delta_4 =$	−2·362 433	−2·462 014
$\beta_{0,2} =$	2·734 807	2·816 395
$\beta_{0,4} =$	0·372 374	0·354 381
$\log \beta_{0,4} =$	9·570 979	9·549 470
$\log (1 : \beta_{0,4}) =$	0·429 021	0·450 530
$\log \phi_4' =$	0·585 487 *n*	0·583 403 *n*
$\log r_4 =$	2·313 172 *n*	2·313 172 *n*
$\log (a_4' - r_4) =$	3·327 680	3·347 105

The values of ϕ_6 and r_6 may now be determined from the formulæ:

$$(14) \qquad \delta_6 = \beta_{4,6} - \delta_4,$$

$$(15) \qquad \phi_6 = \frac{\delta_6}{1 - \text{D}n''},$$

and

$$(16) \qquad r_6 = \frac{(a_4' - d_5)\,\beta_{0,4}}{\phi_6 + \beta_{0,4}},$$

in which only the true focal length for D rays should enter. In the following calculation accordingly only rays of colour D should be considered:—

$$\begin{aligned}
a_4' - r_4 &= 2\,126\cdot572 \\
r_4 &= -\;205\cdot671 \\
a_4' &= 1\,920\cdot901 \\
-d_5 &= -\;5 \\
a_4' - d_5 &= 1\,915\cdot901 \\
\log\,(a_4' - d_5) &= 3\cdot282\,373 \\
\log\,\beta_{0,4} &= 9\cdot570\,979 \\
\log\,(a_4' - d_5)\beta_{0,4} &= 2\cdot853\,352 \\
\beta_{4,6} &= -1\cdot734\,807 \\
-\delta_4 &= 2\cdot362\,433 \\
\delta_6 &= 0\cdot627\,626
\end{aligned}$$

$$\begin{aligned}
\log\,\delta_6 &= 9\cdot797\,701 \\
\log\,[1:(1-n')] &= 0\cdot212\,129n \\
\log\,\phi_6 &= 0\cdot009\,830n \\
\phi_6 &= -1\cdot022\,892 \\
\beta_{0,4} &= 0\cdot372\,374 \\
\phi_6 + \beta_{0,4} &= -0\cdot650\,518 \\
\log\,(\phi_6 + \beta_{0,4}) &= 9\cdot813\,259n \\
\log\,[1:(\phi_6 + \beta_{0,4})] &= 0\cdot186\,741n \\
\log\,(a_4' - d_5)\beta_{0,4} &= 2\cdot853\,352 \\
\log\,r_6 &= 3\cdot040\,093n \\
r_6 &= -1\,096\cdot713
\end{aligned}$$

The value of r_6 being now known, the trigonometrical computation for both colours may be continued from the term $(a_4' - r_4)$ through the complete combination.

		Yellow.	Violet.
$a_4' - r_4$	=	2 126·572	2 223·847
$r_4 - r_6 - d_5$	=	886·042	886·042
$a_4' - r_6 - d_5$	=	3 012·614	3 109·889
$\log\,(a_4' - r_6 - d_5)$	=	3·478 944	3·492 745
$\log\,\beta_{0,4}$	=	9·570 979	9·549 470
$\log\,(1 : r_6)$	=	6·959 907n	6·959 907n
$\log\,\phi_6$	=	0·009 830n	0·002 122n
$\log\,n'$	=	0·207 791	0·215 511
$\log\,\phi_6'$	=	0·217 621n	0·217 633n
ϕ_6	=	−1·022 892	−1·004 898
$-\phi_6'$	=	1·650 521	1·650 567
δ_6	=	0·627 629	0·645 669
$\beta_{0,4}$	=	0·372 374	0·354 381
$\beta_{0,6}$	=	1·000 003	1·000 050
$\log\,\beta_{0,6}$	=	0·000 001	0·000 022
$\log\,(1 : \beta_{0,6})$	=	9·999 999	9·999 978
$\log\,\phi_6'$	=	0·217 621n	0·217 633n
$\log\,r_6$	=	3·040 093n	3·040 093n
$\log\,(a_6' - r_6)$	=	3·257 713	3·257 704

	Yellow.	Violet.
$a_6 - r_6 =$	1 810·143	1 810·106
$r_6 =$	−1 096·713	−1 096·713
$a_6' =$	713·430	713·393

$\log h_0 =$	2·857 333
$\log (1 : \beta_{0,6}) =$	9·999 999
$\log f_{0,6} =$	2·857 332
$f_{0,6} =$	719·999

In accordance with the calculation, the lens combination whose data are as follows:

$h_0 = 30$ mm.
$f_{0,6} = 720$ mm.
$d_1 = 8{\cdot}0$ mm.
$d_3 = 0{\cdot}01$ mm.
$d_5 = 5{\cdot}0$ mm.

$r_0 = +420$ mm.
$r_2 = -200{\cdot}67$ mm.
$r_4 = -205{\cdot}67$ mm.
$r_6 = -1096{\cdot}7$ mm.

has the desired true focal length, namely, $f_{0,6} = 720$, but has still a small chromatic aberration,

$${}_{D}a'_6 - {}_{F}a'_6 = 713{\cdot}430 - 713{\cdot}393 = 0{\cdot}037 \text{ mm.}$$

The reason for the residual chromatic error is that $\beta_{0,2}$ and $\beta_{4,6}$ were determined by means of approximate formulæ, which gave values of the chromatic aberration of the crown and flint lenses of opposite sense, but only approximately equal.

From the calculation it will be seen that the chromatic aberration of the combination is uncorrected, that of the crown lens being the greater. In order to eliminate this error, while retaining the true focal length, the refraction of both lenses must be increased by equal amounts, as a result of which the chromatic aberration of the flint will be more affected than that of the crown lens. The existing error in the objective will consequently be reduced, and it is then a question only of determining by how much the refraction of the two lenses should be increased.

For this purpose there will be used an approximate method which neglects errors of chromatic difference of magnification. As the lens combination for a focal length of 720 mm. has a chromatic aberration of 0·037 mm., the violet ray cuts the axis at a distance of 719·963 mm., and the angle between the two coloured rays for a given common emergence height of 720 mm. is accordingly:

$$\frac{720}{719{\cdot}963} - \frac{720}{720} = 1{\cdot}000\,051\,4'' - 1{\cdot}0'' = 0{\cdot}000\,051\,4''.$$

In the case of the crown glass, for a D ray refraction of $2{\cdot}734\,807''$, the chromatic error is

$$_{\mathrm{V}}\beta_{0,\,2} - {_{\mathrm{D}}\beta_{0,\,2}} = 2{\cdot}816\,395'' - 2{\cdot}734\,807'' = 0{\cdot}081\,588'',$$

and for a refraction of $1{\cdot}734\,807''$ through the flint lens, the error is

$$\begin{aligned}&({_{\mathrm{V}}\beta_{0,\,6}} - {_{\mathrm{V}}\beta_{0,\,2}}) - ({_{\mathrm{D}}\beta_{0,\,6}} - {_{\mathrm{D}}\beta_{0,\,2}})\\ &\quad= (1{\cdot}000\,050'' - 2{\cdot}816\,395'') - (1{\cdot}000\,003'' - 2{\cdot}734\,807'')\\ &\quad= -0{\cdot}081\,541''.\end{aligned}$$

For a deviation of $0{\cdot}001''$, the chromatic aberration of the crown lens is

$$\frac{0{\cdot}081\,588}{2\,734{\cdot}807} = 0{\cdot}000\,029\,87'',$$

and of the flint lens

$$\frac{0{\cdot}081\,541}{1\,734{\cdot}807} = 0{\cdot}000\,047\,00''.$$

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Accordingly, when the refraction of the crown lens and of the flint is increased by $0{\cdot}001''$, the chromatic aberration is altered by an amount

$$0{\cdot}000\,029\,87'' - 0{\cdot}000\,047\,00'' = -0{\cdot}000\,017\,13'',$$

and in the direction of over-correction.

The residual uncorrected chromatic error of the combination, namely $0{\cdot}000\,051\,4''$, may therefore be removed by increasing the refraction of the crown and flint lenses by an amount

$$\frac{0{\cdot}000\,051\,4 \times 0{\cdot}001}{0{\cdot}000\,017\,13} = 0{\cdot}003\,001''.$$

It follows that

$$\beta_{0,\,2} = 2{\cdot}734\,807'' + 0{\cdot}003\,001'' = 2{\cdot}737\,808''$$

and

$$\beta_{4,\,6} = -1{\cdot}734\,807'' - 0{\cdot}003\,001'' = -1{\cdot}737\,808''.$$

Since the value of r_0 must be retained, δ_0 remains unaltered, and

$$\delta_2 = \beta_{0,2} - \delta_0 = 2{\cdot}737\,808'' - 0{\cdot}585\,026'' = 2{\cdot}152\,782''.$$

From the formula

$$(13) \quad r_2 = \frac{(h_0 - d_1\delta_0)({}_{\mathrm{D}}n - 1)}{-\delta_2 + \delta_0({}_{\mathrm{D}}n - 1)}$$

r_2 may be obtained, thus:

$$\begin{aligned}
\delta_0({}_{\mathrm{D}}n - 1) &= 0{\cdot}303\,078 \\
-\delta_2 &= \underline{-2{\cdot}152\,782} \\
-\delta_2 + \delta_0({}_{\mathrm{D}}n - 1) &= -1{\cdot}849\,704 \\
\log\,[-\delta_2 + \delta_0({}_{\mathrm{D}}n - 1)] &= 0{\cdot}267\,124n \\
\log\,(h_0 - d_1\delta_0)({}_{\mathrm{D}}n - 1) &= 2{\cdot}568\,880 \\
\log\,\{1 : [-\delta_2 + \delta_0({}_{\mathrm{D}}n - 1)]\} &= \underline{9{\cdot}732\,876n} \\
\log r_2 &= 2{\cdot}301\,756n \\
r_2 &= -200{\cdot}334\,6
\end{aligned}$$

The radius r_4 should again be made 5 mm. greater than r_2, that is, $r_4 = -205{\cdot}334\,6$ mm.*

Adopting the values thus found, the combination can now be completely computed trigonometrically, the calculation being made as follows for both colours in order to determine whether the chosen combination is completely achromatic or whether there is still present some chromatic aberration.

	Yellow.	Violet.
$a_0' - r_0 =$	810·717	787·164
$r_0 - r_2 - d_1 =$	612·335	612·335
$a_0' - r_2 - d_1 =$	1 423·052	1 399·499
$\log\,(a_0' - r_2 - d_1) =$	3·153 221	3·145 973
$\log \delta_0 =$	9·767 175	9·775 567
$\log\,(1 : r_2) =$	7·698 244n	7·698 244n
$\log \phi_2 =$	0·618 640n	0·619 784n
$\log n =$	0·181 289	0·185 701
$\log \phi_2' =$	0·799 929n	0·805 485n
$\phi_2 =$	−4·155 660	−4·166 621
$-\phi_2' =$	6·308 542	6·389 766
$\delta_2 =$	2·152 882	2·223 145
$\delta_0 =$	0·585 026	0·596 440
$\beta_{0,2} =$	2·737 908	2·819 585

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* See note at foot of p. 90.—ED.

	Yellow.	Violet.
$\log \beta_{0,2} =$	0·437 419	0·450 185
$\log (1 : \beta_{0,2}) =$	9·562 581	9·549 815
$\log \phi_2' =$	0·799 929n	0·805 485n
$\log r_2 =$	2·301 756n	2·301 756n
$\log (a_2' - r_2) =$	2·664 266	2·657 056
$a_2' - r_2 =$	461·600 2	454·000 2
$r_2 - r_4 - d_3 =$	4·99	4·99
$a_2' - r_4 - d_3 =$	466·590 2	458·990 2
$\log (a_2' - r_4 - d_3) =$	2·668 936	2·661 803
$\log \beta_{0,2} =$	0·437 419	0·450 185
$\log (1 : r_4) =$	7·687 538n	7·687 538n
$\log \phi_4 =$	0·793 893n	0·799 526n
$\log (1 : n') =$	9·792 209	9·784 489
$\log \phi_4' =$	0·586 102n	0·584 015n
$\phi_4 =$	−6·221 470	−6·302 692
$-\phi_4' =$	3·855 689	3·837 205
$\delta_4 =$	−2·365 781	−2·465 487
$\beta_{0,2} =$	2·737 908	2·819 585
$\beta_{0,4} =$	0·372 127	0·354 098
$\log \beta_{0,4} =$	9·570 691	9·549 123
$\log (1 : \beta_{0,4}) =$	0·429 309	0·450 877
$\log \phi_4' =$	0·586 102n	0·584 015n
$\log r_4 =$	2·312 462n	2·312 462n
$\log (a_4' - r_4) =$	3·327 873	3·347 354

The values of ϕ_6, δ_6, and r_6 are determined as before, thus:

$a_4' - r_4 = 2\,127{\cdot}516\,7$
$r_4 = -\ 205{\cdot}334\,6$
$a_4' = 1\,922{\cdot}182\,1$
$-d_5 = -\ 5$
$a_4' - d_5 = 1\,917{\cdot}182\,1$

$\log (a_4' - d_5) = 3{\cdot}282\,663$
$\log \beta_{0,4} = 9{\cdot}570\,691$
$\log (a_4' - d_5)\beta_{0,4} = 2{\cdot}853\,354$

$\beta_{4,6} = -1{\cdot}737\,808$
$-\delta_4 = 2{\cdot}365\,781$
$\delta_6 = 0{\cdot}627\,973$

$\log \delta_6 = 9{\cdot}797\,872$
$\log [1 : (1 - {}_{D}n)] = 0{\cdot}212\,129n$
$\log \phi_6 = 0{\cdot}010\,001n$

$\phi_6 = -1{\cdot}023\,295$
$\beta_{0,4} = 0{\cdot}372\,127$
$\phi_6 + \beta_{0,4} = -0{\cdot}651\,168$

$\log (\phi_6 + \beta_{0,4}) = 9{\cdot}813\,693n$
$\log [1 : (\phi_6 + \beta_{0,4})] = 0{\cdot}186\,307n$
$\log (a_4' - d_5)\beta_{0,4} = 2{\cdot}853\,354$
$\log r_6 = 3{\cdot}039\,661n$

$r_6 = -1\,095{\cdot}623$

The computation then proceeds as follows:—

	Yellow.	Violet.
$a_4' - r_4 =$	2 127·517	2 225·123
$r_4 - r_6 - d_5 =$	885·288	885·288
$a_4' - r_6 - d_5 =$	3 012·805	3 110·411
$\log (a_4' - r_6 - d_5) =$	3·478 971	3·492 818
$\log \beta_{0,4} =$	9·570 691	9·549 123
$\log (1 : r_6) =$	6·960 339*n*	6·960 339*n*
$\log \phi_6 =$	0·010 001 *n*	0·002 280*n*
$\log n' =$	0·207 791	0·215 511
$\log \phi_6' =$	0·217 792*n*	0·217 791*n*
$\phi_6 =$	−1·023 295	−1·005 264
$-\phi_6' =$	1·651 171	1·651 167
$\delta_6 =$	0·627 876	0·645 903
$\beta_{0,4} =$	0·372 127	0·354 098
$\beta_{0,6} =$	1·000 003	1·000 001
$\log \beta_{0,6} =$	0·000 001	0·000 000
$\log (1 : \beta_{0,6}) =$	9·999 999	0·000 000
$\log \phi_6' =$	0·217 792*n*	0·217 791*n*
$\log r_6 =$	3·039 661*n*	3·039 661*n*
$\log (a_6' - r_6) =$	3·257 452	3·257 452
$a_6' - r_6 =$	1 809·056	1 809·056
$r_6 =$	−1 095·623	−1 095·623
$a_6' =$	713·433	713·433
$\log h_0 =$	2·857 333	
$\log (1 : \beta_{0,6}) =$	9·999 999	
$\log f_{0,6} =$	2·857 332	
$f_{0,6} =$	719·999	

The calculation indicates that the chosen lens combination is achromatic. It remains to determine how great is the residual spherical aberration. With this in view, there should be traced through the achromatic combination a rim ray of colour D incident at a height $h_0 = 30$ mm.

$$\begin{aligned}\log h_0 &= 1{\cdot}477\,121\\ \log(1:r_0) &= 7{\cdot}376\,751\\ \hline \log\sin\phi_0 &= 8{\cdot}853\,872\\ \log(1:n) &= 9{\cdot}818\,711\\ \hline \log\sin\phi_0' &= 8{\cdot}672\,583\end{aligned}$$

$$\begin{aligned}\phi_0 &= 4^\circ\ 5'\ 45{\cdot}8''\\ -\phi_0' &= -\ 2\ \ 41\ \ 48{\cdot}9\\ \hline \delta_0 &= 1\ \ 23\ \ 56{\cdot}9\end{aligned}$$

$$\begin{aligned}\log\sin\delta_0 &= 8{\cdot}387\,695\\ \log(1:\sin\delta_0) &= 1{\cdot}612\,305\\ \log\sin\phi_0' &= 8{\cdot}672\,583\\ \log r_0 &= 2{\cdot}623\,249\\ \hline \log(a_0'-r_0) &= 2{\cdot}908\,137\end{aligned}$$

$$\begin{aligned}a_0'-r_0 &= 809{\cdot}351\\ r_0-r_2-d_1 &= 612{\cdot}335\\ \hline a_0'-r_2-d_1 &= 1\,421{\cdot}686\end{aligned}$$

$$\begin{aligned}\log(a_0'-r_2-d_1) &= 3{\cdot}152\,804\\ \log\sin\delta_0 &= 8{\cdot}387\,695\\ \log(1:r_2) &= 7{\cdot}698\,244n\\ \hline \log\sin\phi_2 &= 9{\cdot}238\,743n\\ \log n &= 0{\cdot}181\,289\\ \hline \log\sin\phi_2' &= 9{\cdot}420\,032n\end{aligned}$$

$$\begin{aligned}\phi_2 &= -\ 9^\circ\ 58'\ 42{\cdot}4''\\ -\phi_2' &= 15\ \ 15\ \ 3{\cdot}2\\ \hline \delta_2 &= 5\ \ 16\ \ 20{\cdot}8\\ \delta_0 &= 1\ \ 23\ \ 56{\cdot}9\\ \hline \beta_{0,2} &= 6\ \ 40\ \ 17{\cdot}7\end{aligned}$$

$$\begin{aligned}\log\sin\beta_{0,2} &= 9{\cdot}065\,124\\ \log(1:\sin\beta_{0,2}) &= 0{\cdot}934\,876\\ \log\sin\phi_2' &= 9{\cdot}420\,032n\\ \log r_2 &= 2{\cdot}301\,756n\\ \hline \log(a_2'-r_2) &= 2{\cdot}656\,664\end{aligned}$$

$$\begin{aligned}a_2'-r_2 &= 453{\cdot}590\,5\\ r_2-r_4-d_3 &= 4{\cdot}99\\ \hline a_2'-r_4-d_3 &= 458{\cdot}581\end{aligned}$$

$$\begin{aligned}\log(a_2'-r_4-d_3) &= 2{\cdot}661\,416\\ \log\sin\beta_{0,2} &= 9{\cdot}065\,124\\ \log(1:r_4) &= 7{\cdot}687\,538n\\ \hline \log\sin\phi_4 &= 9{\cdot}414\,078n\\ \log(1:n') &= 9{\cdot}792\,209\\ \hline \log\sin\phi_4' &= 9{\cdot}206\,287n\end{aligned}$$

$$\begin{aligned}\phi_4 &= -15^\circ\ 2'\ 17{\cdot}8''\\ -\phi_4' &= 9\ \ 15\ \ 12{\cdot}1\\ \hline \delta_4 &= -\ 5\ \ 47\ \ 5{\cdot}7\\ \beta_{0,2} &= 6\ \ 40\ \ 17{\cdot}7\\ \hline \beta_{0,4} &= 0\ \ 53\ \ 12{\cdot}0\end{aligned}$$

$$\begin{aligned}\log\sin\beta_{0,4} &= 8{\cdot}189\,620\\ \log(1:\sin\beta_{0,4}) &= 1{\cdot}810\,380\\ \log\sin\phi_4' &= 9{\cdot}206\,287n\\ \log r_4 &= 2{\cdot}312\,462n\\ \hline \log(a_4'-r_4) &= 3{\cdot}329\,129\end{aligned}$$

$$\begin{aligned}a_4'-r_4 &= 2\,133{\cdot}678\\ r_4-r_6-d_5 &= 885{\cdot}288\\ \hline a_4'-r_6-d_5 &= 3\,018{\cdot}966\end{aligned}$$

$$\begin{aligned}\log(a_4'-r_6-d_5) &= 3{\cdot}479\,858\\ \log\sin\beta_{0,4} &= 8{\cdot}189\,620\\ \log(1:r_6) &= 6{\cdot}960\,339n\\ \hline \log\sin\phi_6 &= 8{\cdot}629\,817n\\ \log n' &= 0{\cdot}207\,791\\ \hline \log\sin\phi_6' &= 8{\cdot}837\,608n\end{aligned}$$

$$\begin{aligned}\phi_6 &= -\ 2^\circ\ 26'\ 37{\cdot}8''\\ -\phi_6' &= 3\ \ 56\ \ 42{\cdot}9\\ \hline \delta_6 &= 1\ \ 30\ \ 5{\cdot}1\\ \beta_{0,4} &= 0\ \ 53\ \ 12{\cdot}0\\ \hline \beta_{0,6} &= 2\ \ 23\ \ 17{\cdot}1\end{aligned}$$

$$\begin{aligned}
\log \sin \beta_{0,6} &= 8\cdot 619\,801 \\
\log (1 : \sin \beta_{0,6}) &= 1\cdot 380\,199 \\
\log \sin \phi_6' &= 8\cdot 837\,608n \\
\log r_6 &= 3\cdot 039\,661n \\
\log (a_6' - r_6) &= 3\cdot 257\,468
\end{aligned}$$

$$\begin{aligned}
a_6' - r_6 &= 1\,809\cdot 123 \\
r_6 &= -\,1\,095\cdot 623 \\
a_6' &= 713\cdot 500
\end{aligned}$$

Since in the subsequent discussion it is necessary to know the final emergence height, there will be used, for the calculation of the refraction at the last surface, the height formula indicated on p. 77 of Vol. I. This formula also gives a more accurate result, when, as in the previous case, the value of r_6 is considerable.

The calculation is as follows:—

$$\begin{aligned}
\phi_6 &= -2^\circ\ 26'\ 37\cdot 8'' \\
\beta_{0,4} &= 0\ \ 53\ \ 12\cdot 0 \\
\eta_6 = \beta_{0,4} + \phi_6 &= -1\ \ 33\ \ 25\cdot 8 \\
\tfrac{1}{2}\eta_6 &= -0\ \ 46\ \ 42\cdot 9 \\
\log \sin \frac{\eta_6}{2} &= 8\cdot 133\,169n \\
\log \sin^2 \frac{\eta_6}{2} &= 6\cdot 266\,338 \\
\log r_6 &= 3\cdot 039\,661n \\
\log 2 &= 0\cdot 301\,030 \\
\log 2\,r_6 \sin^2 \frac{\eta_6}{2} &= 9\cdot 607\,029n \\
-2r_6 \sin^2 \frac{\eta_6}{2} &= 0\cdot 404\,60 \\
a_4' - d_5 &= 1\,923\cdot 343 \\
a_4' - d_5 - 2r_6 \sin^2 \frac{\eta_6}{2} &= 1\,923\cdot 747\,6
\end{aligned}$$

$$\begin{aligned}
\log \left(a_4' - d_5 - 2r_6 \sin^2 \frac{\eta_6}{2}\right) &= 3\cdot 284\,148 \\
\log \tan \beta_{0,4} &= 8\cdot 189\,672 \\
\log h_6 &= 1\cdot 473\,820 \\
h_6 &= 29\cdot 772\,8 \\
\log h_6 &= 1\cdot 473\,820 \\
\log (1 : \tan \beta_{0,6}) &= 1\cdot 379\,822 \\
\log (h_6 : \tan \beta_{0,6}) &= 2\cdot 853\,642
\end{aligned}$$

$$\begin{aligned} h_6 : \tan \beta_{0,6} &= 713{\cdot}907\,5 \\ 2r_6 \sin^2 \frac{\eta_6}{2} &= 0{\cdot}404\,6 \\ \hline a_6' &= 713{\cdot}502\,9 \end{aligned}$$

It will be seen that the spherical aberration of the lens combination is over-corrected by an amount

$$713{\cdot}433 - 713{\cdot}503 = 0{\cdot}070 \text{ mm.}$$

In order to express this error in angular measure, it should be noted that the rim ray emerges at a height $h_6 = 29{\cdot}773$ mm., and is inclined to the axis at an angle $\beta_{0,6} = 2^\circ\ 23'\ 17{\cdot}1''$. If there is no spherical aberration for the rim ray, this ray will cut the axis in the same place as the axial ray of the same colour. The angle $\beta'_{0,6}$ which this ideal rim ray makes with the axis may be obtained from the formula:

$$(17) \qquad \tan \beta'_{0,6} = \frac{h_6}{\dfrac{h_6}{\tan \beta_{0,6}} - (\text{Spherical Aberration})},$$

where $h_6/\tan \beta_{0,6}$, the projection of the rim ray upon the axis, is in this case 713·9075 mm., and the value of the spherical aberration is 0·070 mm.

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Accordingly,

$$\begin{aligned} \log h_6 &= 1{\cdot}473\,820 \\ \log (1 : 713{\cdot}8375) &= \underline{7{\cdot}146\,401} \\ \log \tan \beta'_{0,6} &= 8{\cdot}620\,221 \\ \beta'_{0,6} &= 2^\circ\ 23'\ 17{\cdot}9'' \end{aligned}$$

Expressed in angular measure, the spherical aberration is therefore

$$2^\circ\ 23'\ 17{\cdot}9'' - 2^\circ\ 23'\ 17{\cdot}1'' = 0{\cdot}8'',$$

and is over-corrected.

This spherical aberration, which is due to the fact that the refraction is not proportional to the angles of incidence and refraction, but to the sines of these angles, will now be corrected by means of an approximate method. The correction will be made by altering the distribution of the refraction over the two surfaces of the flint lens. For the first flint surface the angles of incidence and refraction are respectively

$$\begin{aligned} \phi_4 &= -15^\circ\ \ 2'\ 17{\cdot}8'' \\ \phi_4' &= 9^\circ\ 15'\ 12{\cdot}1'' \end{aligned}$$

The difference between the sines and arcs of these two angles may be obtained from any suitable book of tables.

A short description of the arrangement and use of such tables will be given here. The differences of the values of the arcs and sines may conveniently be tabulated from 0° to 30° in steps of 10″. By the use of the table there can be determined for any refraction the difference arising from the fact that the refraction is proportional to the sine and not to the angle itself.

In the use of the tables it is assumed that the trigonometrical computation of the objective is so performed that the spherical aberration can be expressed in angular measure. It is necessary in such a case to find the inclination of two lines, both of which proceed from the emergence point of the ray at the last surface. Of these one is directed towards the point of intersection of the paraxial ray, and the other towards the point of intersection of the rim ray of the same colour with the axis. (The latter is accordingly the actual rim ray.) If the angle which the line drawn through the point of intersection of the paraxial ray makes with the axis is greater than the angle which the rim ray makes with the axis, the spherical aberration is over-corrected, and vice versa. The form of the last lens must then be altered; that is, the deviation must be differently distributed over the two surfaces, while retaining the same total refraction. In this connection only the largest angle that occurs need be considered. When the spherical aberration is over-corrected, the greatest angle of incidence must be made smaller if it corresponds with a refraction *from the axis*. On the other hand, if it corresponds with a refraction *towards the axis*, it must be made correspondingly greater.

In the example already computed, the spherical aberration was 0·8″, and was over-corrected. The largest angle occurring in the flint lens, namely −15° 2′ 17·8″, corresponded with a refraction from the axis. From tables it will be seen that for an angle of 15° 2′ 10″ the difference is 10′ 19·19″ = 619·19″. For an angle of 15° 2′ 20″ the difference is 10′ 19·53″ = 619·53″. For the angle in question the difference is accordingly 10′ 19·46″ = 619·46″.

The corresponding angle of emergence is +9° 15′ 12·1″. From tables the difference for an angle of 9° 15′ 10″ = 2′ 24·60″, and for an angle 9° 15′ 20″ = 2′ 24·73″. For the angle in question the difference is therefore 2′ 24·63″ = 144·63″

The difference in the deviation of the largest angle, namely 619·46″, is greater than the difference of the total deviation (474·83″) at the first surface of this lens, in the ratio $\frac{619{\cdot}46''}{474{\cdot}83''}$. In order to eliminate the spherical aberration, the difference of the deviation must be reduced by 0·8″. Accordingly that of the largest angle must be reduced in the ratio of $\frac{619{\cdot}46}{474{\cdot}83}$ to 0·8″; that is, by an amount 1·04″. Thus the largest angle should have a difference of 619·46″ − 1·04″ = 618·42″.

From tables it will be found that for an angle of 15° 1′ 40″ the difference is 10′ 18·15″, and for an angle of 15° 1′ 50″ the difference is 10° 18′ 49″. Therefore the difference of 618·42″ corresponds with an angle of 15° 1′ 48″.

It has been found that the difference between the sine and the arc for the first angle, namely −15° 2′ 17·8″, = 619·46″, and for the second angle, +9° 15′ 12·1″, = 144·63″.

Also the ratio of the difference for the whole refraction to the difference for the first angle is $\frac{474{\cdot}83}{619{\cdot}46} = 1 : 1{\cdot}304$.

In order to eliminate the whole spherical aberration of 0·8″, by an alteration of the first angle, it is necessary that the difference between the sine and the arc of this angle should have a less effect by an amount 1·04″ than it had for an angle −15° 2′ 17·8″; that is, it should be 618·42″, and ϕ_4 must now be taken as −15° 1′ 48″. By means of this value, r_4 may then be determined from the formula already used on several occasions, thus:

$$\begin{aligned}
\phi_4 &= -15^\circ\ 1'\ 48'' \\
\log\sin\phi_4 &= 9{\cdot}413\,844n \\
\sin\phi_4 &= -0{\cdot}259\,324 \\
\log\sin\beta_{0,2} &= 9{\cdot}065\,124 \\
\sin\beta_{0,2} &= 0{\cdot}116\,178 \\
\sin\phi_4 &= \underline{-0{\cdot}259\,324} \\
\sin\beta_{0,2} + \sin\phi_4 &= -0{\cdot}143\,146 \\
\log(\sin\beta_{0,2} + \sin\phi_4) &= 9{\cdot}155\,780n \\
\log(a_2' - d_3) &= 2{\cdot}403\,543 \\
\log\sin\beta_{0,2} &= 9{\cdot}065\,124 \\
\log[1 : (\sin\beta_{0,2} + \sin\phi_4)] &= \underline{0{\cdot}844\,220n} \\
\log r_4 &= 2{\cdot}312\,887n \\
r_4 &= -205{\cdot}535\,7
\end{aligned}$$

In order to see whether the approximate method used has served to remove the spherical aberration completely, there should be traced through the flint lens the paraxial and rim rays from the point of their emergence from the crown lens, the value of r_4 just determined being employed.

The computation is as follows:—

	Paraxial Ray.	Rim Ray.
$a_2' - r_2 =$	461·600 2	453·590 5
$r_2 - r_4 - d_3 =$	5·191 1	5·191 1
$a_2' - r_4 - d_3 =$	466·791 3	458·781 6
$\log (a_2' - r_4 - d_3) =$	2·669 123	2·661 606
$\log \sin \beta_{0,2} =$	0·437 419	9·065 124
$\log (1 : r_4) =$	7·687 113n	7·687 113n
$\log \sin \phi_4 =$	0·793 655n	9·413 843n
$\log (1 : n') =$	9·792 209	9·792 209
$\log \sin \phi_4' =$	0·585 684n	9·206 052n
$\phi_4 =$	−6·218 054	−15° 1′ 47·8″
$-\phi_4' =$	3·853 572	9 14 53·9
$\delta_4 =$	−2·364 482	− 5 46 53·9
$\beta_{0,2} =$	2·737 908	6·40 17·7
$\beta_{0\,4} =$	0·373 426	0 53 23·8
$\log \sin \beta_{0,4} =$	9·572 205	8·191 216
$\log (1 : \sin \beta_{0,4}) =$	0·427 795	1·808 784
$\log \sin \phi_4' =$	0·585 864n	9·206 052n
$\log r_4 =$	2·312 887n	2·312 887n
$\log (a_4' - r_4) =$	3·326 546	3·327 723

For the determination of r_6 only the paraxial D ray need be considered.

$a_4' - r_4 =$	2 121·027
$r_4 =$	− 205·536
$a_4' =$	1 915·491
$-d_5 =$	− 5
$a_4' - d_5 =$	1 910·491

$\log (a_4' - d_5) =$ 3·281 145
$\log \beta_{0,4} =$ 9·572 205
$\log (a_4' - d_5)\, \beta_{0,4} =$ 2·853 350

$\beta_{4,6} =$ −1·737 808
$-\delta_4 =$ 2·364 482
$\delta_6 =$ 0·626 674

$\log \delta_6 =$ 9·796 972
$\log [1 : (1 - {}_{D}n)] =$ 0·212 129n
$\log \phi_6 =$ 0·009 101n

$\phi_6 =$ −1·021 178
$\beta_{0,4} =$ 0·373 426
$\phi_6 + \beta_{0,4} =$ −0·647 752

$\log (\phi_6 + \beta_{0,4}) =$ 9·811 409n
$\log [1 : (\phi_6 + \beta_{0,4})] =$ 0·188 591n
$\log (a_4' - d_5)\, \beta_{0,4} =$ 2·853 350
$\log r_6 =$ 3·041 941n

$r_6 =$ −1 101·389 1

For the remainder of the computation the paraxial and rim rays should both be considered.

	Paraxial Ray.	Rim Ray.
$a_4' - r_4 =$	2 121·027	2 126·782
$r_4 - r_6 - d_5 =$	890·853	890·853
$a_4' - r_6 - d_5 =$	3 011·880	3 017·635
$\log (a_4' - r_6 - d_5) =$	3·478 838	3·479 667
$\log \sin \beta_{0,4} =$	9·572 205	8·191 216
$\log (1 : r_6) =$	6·958 059n	6·958 059n
$\log \sin \phi_6 =$	0·009 102n	8·628 942n
$\log n' =$	0·207 791	0·207 791
$\log \sin \phi_6' =$	0·216 893n	8·836 733n
$\phi_6 =$	−1·021 178	−2° 26′ 20·1″
$-\phi_6' =$	1·647 754	3 56 14·3
$\delta_6 =$	0·626 576	1 29 54·2
$\beta_{0,4} =$	0·373 426	0 53 23·7
$\beta_{0,6} =$	1·000 002	2 23 17·9
$\log \sin \beta_{0,6} =$	0·000 001	8·619 843
$\log (1 : \sin \beta_{0,6}) =$	9·999 999	1·380 157
$\log \sin \phi_6' =$	0·216 892n	8·836 733n
$\log r_6 =$	3·041 941n	3·041 941n
$\log (a_6' - r_6) =$	3·258 832	3·258 831
$a_6' - r_6 =$	1 814·814 2	1 814·808 4
$r_6 =$	−1 101·389 1	−1 101·389 1
$a_6' =$	713·425 1	713·419 3

Example of the Further Correction of Spherical Aberration

Although the spherical aberration is uncorrected by an amount of 0·006 mm., which is smaller than the amount permissible under the conditions specified, an attempt will be made to reduce the error still further, merely to illustrate all that is involved. This will be done by determining a still better value of ϕ_4 by interpolation.

For an angle of incidence, $\phi_4 = -15^\circ\ 2'\ 17{\cdot}8''$, the spherical aberration was over-corrected by an amount 0·070 mm. For an angle of $-15^\circ\ 1'\ 48''$ the error was uncorrected by an amount 0·006 mm. Accordingly there should be no spherical aberration corresponding with an angle of incidence of $-15^\circ\ 1'\ 50{\cdot}4''$.

It should be noted that in this particular example the residual error is almost within the limits of accuracy of the computation, and that, therefore, as the result of interpolation a much better value cannot be obtained. A somewhat more proportionate value of the angle of incidence will accordingly be assumed, namely $\phi_4 = -15^\circ\ 1'\ 49{\cdot}5''$, and in the first instance this angle of incidence will be used for the determination of the radius r_4.

$$
\begin{aligned}
\phi_4 &= -15^\circ\ 1'\ 49{\cdot}5'' \\
\log \sin \phi_4 &= 9{\cdot}413\,856n \\
\sin \phi_4 &= -0{\cdot}259\,332 \\
\log \sin \beta_{0,2} &= 9{\cdot}065\,124 \\
\sin \beta_{0,2} &= 0{\cdot}116\,178 \\
\sin \phi_4 &= \underline{-0{\cdot}259\,332} \\
\sin \beta_{0,2} + \sin \phi_4 &= -0{\cdot}143\,154 \\
\log (\sin \beta_{0,2} + \sin \phi_4) & \\
&= 9{\cdot}155\,803n
\end{aligned}
$$

$$
\begin{aligned}
\log (a_2' - d_3) &= 2{\cdot}403\,542 \\
\log \sin \beta_{0,2} &= 9{\cdot}065\,124 \\
\log [1 : (\sin \beta_{0,2} + \sin \phi_4)] & \\
&= \underline{0{\cdot}844\,197n} \\
\log r_4 &= 2{\cdot}312\,864n \\
r_4 &= -205{\cdot}524\,8
\end{aligned}
$$

It is now possible to compute for the flint lens the yellow and violet paraxial rays, and also the yellow rim ray, as follows:—

	Paraxial Ray: Yellow.	Paraxial Ray: Violet.	Rim Ray: Yellow.
$a_2' - r_2 =$	461·600 2	454·000 2	453·590 5
$r_2 - r_4 - d_3 =$	5·180 2	5·180 2	5·180 2
$a_2' - r_4 - d_3 =$	466·780 4	459·180 4	458·770 7
$\log (a_2'-r_4-d_3) =$	2·669 113	2·661 983	2·661 596
$\log \sin \beta_{0,2} =$	9·437 419	9·450 185	9·065 124
$\log (1 : r_4) =$	7·687 136*n*	7·687 136*n*	7·687 136*n*
$\log \sin \phi_4 =$	9·793 667*n*	9·799 304*n*	9·413 856*n*
$\log (1 : n') =$	9·792 209	9·784 489	9·792 209
$\log \sin \phi_4' =$	9·585 876*n*	9·583 793*n*	9·206 065*n*
$\phi_4 =$	−6·218 239	−6·299 473	−15° 1′ 49·5″
$-\phi_4' =$	3·853 687	3·835 246	9 14 54·9
$\delta_4 =$	−2·364 552	−2·464 227	− 5 46 54·6
$\beta_{0,2} =$	2·737 908	2·819 585	6 40 17·7
$\beta_{0,4} =$	0·373 356	0·355 358	0 53 23·1
$\log \sin \beta_{0,4} =$	9·572 123	9·550 667	8·191 128
$\log (1 : \sin \beta_{0,4}) =$	0·427 877	0·449 333	1·808 872
$\log \sin \phi_4' =$	0·585 876*n*	0·583 793*n*	9·206 065*n*
$\log r_4 =$	2·312 864*n*	2·312 864*n*	2·312 864*n*
$\log (a_4' - r_4) =$	3·326 617	3·345 991	3·327 801

The radius r_6 will now be calculated by means of the yellow paraxial ray.

$a_4' - r_4 =$ 2 121·374 1
$r_4 =$ − 205·524 8
$a_4' =$ 1 915·849 3
$-d_5 =$ − 5
$a_4' - d_5 =$ 1 910·849 3

$\log (a_4' - d_5) =$ 3·281 226
$\log \beta_{0,4} =$ 9·572 123
$\log (a_4'-d_5)\,\beta_{0,4} =$ 2·853 349

$\beta_{4,6} =$ −1·737 908
$-\delta_4 =$ 2·364 552
$\delta_6 =$ 0·626 644

$\log \delta_6 =$ 9·797 021
$\log [1 : (1 - {}_{D}n)] =$ 0·212 129*n*
$\log \phi_6 =$ 0·009 150*n*

$\phi_6 =$ −1·021 292
$\beta_{0,4} =$ 0·373 356
$\phi_6 + \beta_{0,4} =$ −0·647 936

$\log (\phi_6 + \beta_{0,4}) =$ 9·811 532*n*
$\log [1 : (\phi_6+\beta_{0,4})] =$ 0·188 468*n*
$\log (a_4'-d_5)\,\beta_{0,4} =$ 2·853 350
$\log r_6 =$ 3·041 818*n*

$r_6 =$ −1 101·077

Using the same three rays the computation is continued as follows:—

	Paraxial Ray.		Rim Ray:
	Yellow.	Violet.	Yellow.
$a_4' - r_4 =$	2 121·374 1	2 218·149 0	2 127·164 7
$r_4 - r_6 - d_5 =$	890·552 4	890·552 4	890·552 4
$a_4' - r_6 - d_5 =$	3 011·926 5	3 108·701 4	3 017·717 1
$\log (a_4' - r_6 - d_5) =$	3·478 844	3·492 579	3·479 679
$\log \sin \beta_{0,4} =$	9·572 123	9·550 667	8·191 128
$\log (1 : r_6) =$	6·958 182*n*	6·958 182*n*	6·958 182*n*
$\log \sin \phi_6 =$	0·009 150*n*	0·001 428*n*	8·628 989*n*
$\log n' =$	0·207 791	0·215 511	0·207 791
$\log \sin \phi_6' =$	0·216 941*n*	0·216 939*n*	8·836 780*n*
$\phi_6 =$	−1·021 292	−1·003 293	− 2° 26′ 21·0″
$-\phi_6' =$	1·647 938	1·647 930	3 56 15·8
$\delta_6 =$	0·626 646	0·644 637	1 29 54·8
$\beta_{0,4} =$	0·373 356	0·355 359	0 53 23·1
$\beta_{0,6} =$	1·000 002	0·999 996	2 23 17·9
$\log \sin \beta_{0\ 6} =$	0·000 001	9·999 998	8·619 839
$\log (1 : \sin \beta_{0\ 6}) =$	9·999 999	0·000 002	1·380 161
$\log \sin \phi_6' =$	0·216 941*n*	0·216 939*n*	8·836 780*n*
$\log r_6 =$	3·041 818*n*	3·041 818*n*	3·041 818*n*
$\log (a_6' - r_6) =$	3·258 758	3·258 759	3·258 759
$a_6' - r_6 =$	1 814·503 3	1 814·506 7	1 814·507 1
$r_6 =$	−1 101·077 2	−1 101·077 2	−1 101·077 2
$a_6' =$	713·426 1	713·429 5	713·429 9
$\log h_0 =$	2·857 333	2·857 333	1·477 121
$\log (1 : \sin \phi_6') =$	9·999 999	0·000 002	1·380 161
$\log f_{0,6} =$	2·857 332	2·857 335	2·857 282
$f_{0,6} =$	719·999	720·003	719·917

The violet rim ray should now be calculated through the whole combination.

$$\begin{array}{rcl}
\log h_0 & = & 1\cdot477\,121 \\
\log (1 : r_0) & = & 7\cdot376\,751 \\ \hline
\log \sin \phi_0 & = & 8\cdot853\,872 \\
\log (1 : n) & = & 9\cdot814\,299 \\ \hline
\log \sin \phi_0' & = & 8\cdot668\,171
\end{array}$$

$$\begin{array}{rcl}
\phi_0 & = & 4^\circ\ 5'\ 45\cdot8'' \\
-\phi_0' & = & -\ 2\ 40\ 10\cdot7 \\ \hline
\delta_0 & = & 1\ 25\ 35\cdot1
\end{array}$$

$$\begin{array}{rcl}
\log \sin \delta_0 & = & 8\cdot396\,079 \\
\log (1 : \sin \delta_0) & = & 1\cdot603\,921 \\
\log \sin \phi_0' & = & 8\cdot668\,171 \\
\log r_0 & = & 2\cdot623\,249 \\ \hline
\log (a_0' - r_0) & = & 2\cdot895\,341
\end{array}$$

$$\begin{array}{rcl}
a_0' - r_0 & = & 785\cdot852\,9 \\
r_0 - r_2 - d_1 & = & 612\cdot334\,6 \\ \hline
a_0' - r_2 - d_1 & = & 1\,398\cdot187\,5
\end{array}$$

$$\begin{array}{rcl}
\log (a_0' - r_2 - d_1) & = & 3\cdot145\,565 \\
\log \sin \delta_0 & = & 8\cdot396\,079 \\
\log (1 : r_2) & = & 7\cdot698\,244n \\ \hline
\log \sin \phi_2 & = & 9\cdot239\,888n \\
\log n & = & 0\cdot185\,701 \\ \hline
\log \sin \phi_2' & = & 9\cdot425\,589n
\end{array}$$

$$\begin{array}{rcl}
\phi_2 & = & -10^\circ\ 0'\ 18\cdot3'' \\
-\phi_2' & = & 15\ 27\ 7\cdot8 \\ \hline
\delta_2 & = & 5\ 26\ 49\cdot5 \\
\delta_0 & = & 1\ 25\ 35\cdot1 \\ \hline
\beta_{0,2} & = & 6\ 52\ 24\cdot6
\end{array}$$

$$\begin{array}{rcl}
\log \sin \beta_{0,2} & = & 9\cdot078\,013 \\
\log (1 : \sin \beta_{0,2}) & = & 0\cdot921\,987 \\
\log \sin \phi_2' & = & 9\cdot425\,589n \\
\log r_2 & = & 2\cdot301\,756n \\ \hline
\log (a_2' - r_2) & = & 2\cdot649\,332
\end{array}$$

$$\begin{array}{rcl}
a_2' - r_2 & = & 445\cdot997\,1 \\
r_2 - r_4 - d_3 & = & 5\cdot180\,2 \\ \hline
a_2' - r_4 - d_3 & = & 451\cdot177\,3
\end{array}$$

$$\begin{array}{rcl}
\log (a_2' - r_4 - d_3) & = & 2\cdot654\,347 \\
\log \sin \beta_{0,2} & = & 9\cdot078\,013 \\
\log (1 : r_4) & = & 7\cdot687\,136n \\ \hline
\log \sin \phi_4 & = & 9\cdot419\,496n \\
\log (1 : n') & = & 9\cdot784\,489 \\ \hline
\log \sin \phi_4' & = & 9\cdot203\,985n
\end{array}$$

$$\begin{array}{rcl}
\phi_4 & = & -15^\circ\ 13'\ 53\cdot9'' \\
-\phi_4' & = & 9\ 12\ 14\cdot5 \\ \hline
\delta_4 & = & -\ 6\ 1\ 39\cdot4 \\
\beta_{0,2} & = & 6\ 52\ 24\cdot6 \\ \hline
\beta_{0,4} & = & 0\ 50\ 45\cdot2
\end{array}$$

$$\begin{array}{rcl}
\log \sin \beta_{0,4} & = & 8\cdot169\,175 \\
\log (1 : \sin \beta_{0,4}) & = & 1\cdot830\,825 \\
\log \sin \phi_4' & = & 9\cdot203\,985n \\
\log r_4 & = & 2\cdot312\,864n \\ \hline
\log (a_4' - r_4) & = & 3\cdot347\,675
\end{array}$$

$$\begin{array}{rcl}
a_4' - r_4 & = & 2\,226\cdot766\,2 \\
r_4 - r_6 - d_5 & = & 890\cdot552\,4 \\ \hline
a_4' - r_6 - d_5 & = & 3\,117\cdot318\,6
\end{array}$$

$$\begin{array}{rcl}
\log (a_4' - r_6 - d_5) & = & 3\cdot493\,781 \\
\log \sin \beta_{0,4} & = & 8\cdot169\,175 \\
\log (1 : r_6) & = & 6\cdot958\,182n \\ \hline
\log \sin \phi_6 & = & 8\cdot621\,138n \\
\log n' & = & 0\cdot215\,511 \\ \hline
\log \sin \phi_6' & = & 8\cdot836\,649n
\end{array}$$

$$\begin{array}{rcl}
\phi_6 & = & -\ 2^\circ\ 23'\ 43\cdot6'' \\
-\phi_6' & = & 3\ 56\ 11\cdot5 \\ \hline
\delta_6 & = & 1\ 32\ 27\cdot9 \\
\beta_{0,4} & = & 0\ 50\ 45\cdot2 \\ \hline
\beta_{0,6} & = & 2\ 23\ 13\cdot1
\end{array}$$

$$\begin{aligned}
\log \sin \beta_{0,6} &= 8{\cdot}619\,598 \\
\log (1 : \sin \beta_{0,6}) &= 1{\cdot}380\,402 \\
\log \sin \phi_6' &= 8{\cdot}836\,649n \\
\log r_6 &= 3{\cdot}041\,818n \\
\hline
\log (a_6' - r_6) &= 3{\cdot}258\,869
\end{aligned}$$

$$\begin{aligned}
a_6' - r_6 &= 1\,814{\cdot}968\,6 \\
r_6 &= -\,1\,101{\cdot}077\,2 \\
\hline
a_6' &= 713{\cdot}891\,4
\end{aligned}$$

$$\begin{aligned}
\log h_0 &= 1{\cdot}477\,121 \\
\log (1 : \sin \beta_{0,6}) &= 1{\cdot}380\,402 \\
\hline
\log f_{0,6} &= 2{\cdot}857\,523
\end{aligned}$$

$$f_{0,6} = 720{\cdot}316$$

The results of the computation may be summarized as follows:—

The data for the combination were:

mm.

$r_0 = +\ 420{\cdot}00$

$d_1 = 8{\cdot}0$

$r_2 = -\ 200{\cdot}33$

$d_3 = 0{\cdot}01$

$r_4 = -\ 205{\cdot}52$

$d_5 = 5{\cdot}0$

$r_6 = -\ 1\,101{\cdot}10$

The back focal lengths are:

mm.

$({}_{\mathrm{D}}a_6')$ axis $= 713{\cdot}43$

$({}_{\mathrm{V}}a_6')$ axis $= 713{\cdot}43$

$({}_{\mathrm{D}}a_6')$ rim $= 713{\cdot}43$

$({}_{\mathrm{V}}a_6')$ rim $= 713{\cdot}89$

The focal lengths are:

mm.

$({}_{\mathrm{D}}f_{0,6})$ axis $= 720{\cdot}00$

$({}_{\mathrm{V}}f_{0,6})$ axis $= 720{\cdot}00$

$({}_{\mathrm{D}}f_{0,6})$ rim $= 719{\cdot}92$

$({}_{\mathrm{V}}f_{0,6})$ rim $= 720{\cdot}32$

Further, the computation gives the following important angles, namely:

The total deviation of the combination:

$$({}_D\beta_{0,6})_{\text{rim}} = 2^\circ\ 23'\ 18''$$

The deviations of the individual lenses:

$$\delta_0 + \delta_2 = +6^\circ\ 40'\ 18''$$
$$\delta_4 + \delta_6 = -4\ \ 17\ \ 0$$

The deviations at the individual surfaces:

$$\delta_0 = +1^\circ\ 23'\ 57''$$
$$\delta_2 = +5\ \ 16\ \ 21$$
$$\delta_4 = -5\ \ 46\ \ 55$$
$$\delta_6 = +1\ \ 29\ \ 55$$

Finally, from the back focal lengths and the focal lengths, the errors in the image formed by the lens combination may be calculated. They are as follows:—

		mm.
$({}_Da_6')_{\text{axis}} - ({}_Va_6')_{\text{axis}}$	=	0·00
$({}_Da_6')_{\text{axis}} - ({}_Da_6')_{\text{rim}}$	=	0·00
$({}_Df_{0,6})_{\text{axis}} - ({}_Vf_{0,6})_{\text{axis}}$	=	0·00
$({}_Df_{0,6})_{\text{axis}} - ({}_Df_{0,6})_{\text{rim}}$	=	+0·08
$({}_Da_6')_{\text{rim}} - ({}_Va_6')_{\text{rim}}$	=	−0·46
$({}_Df_{0,6})_{\text{rim}} - ({}_Vf_{0,6})_{\text{rim}}$	=	−0·40

Of the above, the first error is the chromatic aberration, the second the spherical aberration, the third the chromatic difference of magnification, the fourth the error against the sine condition, the fifth the Gauss-condition error, and the last the chromatic difference of magnification for the rim ray. (See Vol. I, pp. 52–54.)

CHAPTER III

Achromatic Doublet Objectives—*Cont.*

Tables of Achromatic Doublet Objectives corrected for Spherical Aberration

In the same manner as for the example just described, there has been computed a series of achromatic objectives in which the spherical aberration has also been corrected. There has thus been obtained a table of doublet objectives which indicates the values of the residual image errors in the various forms of objectives dealt with. For this purpose the objectives have been divided into two principal groups, in the first of which the preceding lens is the crown and in the other the flint. These principal groups, which have been denoted by the capital letters A and B, have each been divided into two sub-groups, (*a*) and (*b*), in which the refractions at the two surfaces of the back lens are distributed in opposite manners. There are accordingly four individual groups, A (*a*); A (*b*); B (*a*); and B (*b*). The various examples in the individual groups are arranged, according to the refraction at the first surface of the front lens and in steps of 2°, with respect to the objective computed above whose first radius is 420·00 mm., and in which the deviation at the first surface for a semi-aperture of 30 mm. is $1^\circ\ 23'\ 57''$. In addition there are given six other objectives in which the respective refractions at the first surfaces are as follows: $+9^\circ\ 23'\ 57''$, $+7^\circ\ 23'\ 57''$, $+5^\circ\ 23'\ 57''$, $+3^\circ\ 23'\ 57''$, $-0^\circ\ 36'\ 3''$, and $-2^\circ\ 36'\ 3''$.

In Table A (*a*), pp. 118 and 119, it is assumed that the refraction at the first surface of the flint lens is in a direction away *from* the axis, and at the second surface *towards* the axis, whereas in Table A (*b*), pp. 122 and 123, the refraction away *from* the axis takes place at the second surface of the flint lens.

In Table B (*a*), pp. 126 and 127, it is assumed that the principal refraction of the crown lens is towards the axis, and that the refraction at the second face is smaller, and according to the requirements, is directed either towards or from the axis, as the case may be.

The examples given in this table are for values of $\delta_0 = +7^\circ\,38'\,40''$, $+5^\circ\,38'\,40''$, $+3^\circ\,38'\,40''$, $+1^\circ\,38'\,40''$, and $-6^\circ\,21'\,20''$. The data for objectives having values of $\delta_0 = -0^\circ\,21'\,20''$, $-2^\circ\,21'\,20''$, and $-4^\circ\,21'\,20''$ must be omitted, since in these cases the spherical aberration cannot be eliminated. Finally, in Table B (*b*), pp. 118 and 119, the stronger refraction of the crown glass occurs at the second surface.

The numbering of the lenses in the individual tables is not consecutive, as there have been omitted various numbers which are allocated to certain special lenses described later.

Table A (*a*) commences with the lens, Case 1, in which the strongest refraction, namely $+9^\circ\,23'\,57''$, takes place at the first surface. This refraction, as already mentioned, decreases in the subsequent lenses by steps of 2° so that in the Case 5*a* the refractions at the two surfaces of the crown lens are nearly equal.

Since from Case 1*a* to Case 5*a* the distribution of the refraction over the two surfaces of the crown lens becomes progressively more equal, the spherical aberration of these crown lenses must become gradually smaller. Similarly, the distribution of the refractions over the two surfaces of the flint lenses becomes more equal, the initial values being $-9^\circ\,7'\,46''$ *from* the axis at the first surface, and $+3^\circ\,6'\,15''$ *towards* the axis at the second surface, and, in Case 5*a*, $-4^\circ\,16'\,4''$ and $-0^\circ\,3'\,35''$, both refractions being away *from* the axis. Progressively weaker lenses are thus involved.

After Case 5*a* the refractions at the two surfaces of the crown lens become more unequal again, and the spherical aberration increases. In consequence of this, the distribution of the refractions over the two surfaces of the corresponding flint lens must be more unequal, the lenses thus becoming more curved.

Proceeding from Case 5*a* towards the left, the distribution becomes progressively more unequal, the refraction of the first surface increasing. Proceeding towards the right, the distribution also becomes more unequal, but the increase of refraction takes place at the second surface. Similarly, proceeding from left to right, the flint lenses up to Case 5 *a* become progressively flatter and thereafter again more curved.

A comparison of the values of the refractions δ_0, δ_2, δ_4, and δ_6, in conjunction with the consideration of the diagrams of the objective sections, indicates clearly the manner in which the forms of the lenses change.

In all the objectives of this table a true focal length of 720 mm. has been obtained and the spherical aberration removed. As a result of this, the back focal length, that is, the distance of the focal point from the vertex of the last surface, varies in the sense that through Case 5 *a* there is a gradual increase for the yellow axial ray. In Case 1 *a* the back focal length is 49·50 mm. shorter than the true focal length. Proceeding towards the right of the table, it increases progressively. Between 11 *a* and 18 *a* there must accordingly be one case where the back focal length and the true focal length are equal, whereas in Case 19 *a* the back focal length is 16·26 mm. greater than the true focal length.

Corresponding with the alteration in the back focal length, there is an alteration in the total deviation of the rim ray. Reading from left to right, the total deviation becomes progressively smaller, from a value of 2° 30′ 34″ to 2° 22′ 34″, the reason being that the emergence height decreases from left to right. Further, it will be seen that there are differences in the distribution of the refraction over the two lenses. Thus the refraction of both the crown and the flint lenses is a maximum in Case 1 *a*, diminishes to Case 11 *a*, and thereafter again increases. With regard to the image aberrations, it is desirable, as mentioned previously, that they should be removed as far as possible, and should in no case exceed 0·05 mm. Thereby the condition is always fulfilled that the chromatic aberration at the axis and the spherical aberration for the D rays should be eliminated. In addition the tables indicate other three errors, namely, the chromatic aberration at the rim when the corresponding error at the axis is zero (the so-called Gauss condition), the error in the position of the principal points for the two colours in the axis, and the similar error for the axial and rim rays of colour D. In all cases included in Table A (*a*), the chromatic aberration at the rim is of such a nature that the more strongly refracted ray has a longer back focal length; that is, $({}_{D}a'_{6})_{\text{rim}} - ({}_{V}a'_{6})_{\text{rim}}$ is negative. In Case 1 *a* this error has its greatest value −9·17 mm. It then diminishes to −0·46 mm. in Case 11 *a*, and thereafter again increases to a value −2·80 mm., in Case 19 *a*. In Case 1 *a* the errors at the principal points reach their greatest values of −1·13 mm. and +34·80 mm. respectively. They then diminish until, in Case 11 *a*, they become zero and +0·08 mm. respectively. Thereafter they increase again, while both signs change, to the values +0·52 mm. and −3·65 mm.

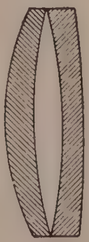
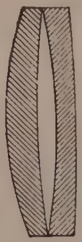
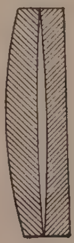
It should be noted in this connection that an error at the principal points for the two colours gives rise to images of unequal size for these two colours, and that an error in the principal points between the axial and rim rays indicates a departure from the sine condition.

The difference in the principal points for the two colours has not been indicated for the rim of the objective, as the variations are essentially the same as those of the back focal lengths for the rim rays of these colours.

(*a*) THE STRONGER REFRACTION AT THE

$f_{0,6} = 720$ mm.;

${}_{\mathrm{D}}n = 1{\cdot}518\,06$; ${}_{\mathrm{V}}n = 1{\cdot}533\,56$;

Particulars.	Case 1*a*.	Case 2*a*.	Case 4*a*.
r_0	+ 67·282	+ 83·159	+ 111·50
d_1	8·0	8·0	8·0
r_2	+ 153·02	+ 244·88	+ 764·18
d_3	6·685	4·44	2·121
r_4	− 114·88	− 164·62	− 285·95
d_5	5·0	5·0	5·0
r_6	− 300·33	− 783·79	+ 1132·1
$({}_{\mathrm{D}}a_6')$ axis	670·50	681·54	692·02
$({}_{\mathrm{V}}a_6')$ axis	670·50	681·50	691·98
$({}_{\mathrm{D}}a_6')$ rim	670·46	681·54	692·03
$({}_{\mathrm{V}}a_6')$ rim	679·63	686·71	694·04
$({}_{\mathrm{D}}f_{0,6})$ axis	720·00	720·00	720·00
$({}_{\mathrm{V}}f_{0,6})$ axis	721·13	720·77	720·48
$({}_{\mathrm{D}}f_{0,6})$ rim	685·20	697·41	708·17
$({}_{\mathrm{V}}f_{0,6})$ rim	694·48	702·75	710·49
$({}_{\mathrm{D}}\beta_{0,6})$ rim	$2^\circ\ 30'\ 34''$	$2^\circ\ 27'\ 55''$	$2^\circ\ 25'\ 40''$
$\delta_0 + \delta_2$	+ 8 32 5	+ 7 39 11	+ 7 3 1
$\delta_4 + \delta_6$	− 6 1 31	− 5 11 16	− 4 37 21
δ_0	+ 9 23 57	+ 7 23 57	+ 5 23 57
δ_2	− 0 51 52	+ 0 15 14	+ 1 39 4
δ_4	− 9 7 46	− 6 56 37	− 4 58 53
δ_6	+ 3 6 15	+ 1 45 21	+ 0 21 32
$({}_{\mathrm{D}}a_6')$ axis − $({}_{\mathrm{V}}a_6')$ axis	0·00	+ 0·04	+ 0·04
$({}_{\mathrm{D}}a_6')$ axis − $({}_{\mathrm{D}}a_6')$ rim	+ 0·04	0·00	− 0·01
$({}_{\mathrm{D}}f_{0,6})$ axis − $({}_{\mathrm{V}}f_{0,6})$ axis	− 1·13	− 0·77	− 0·48
$({}_{\mathrm{D}}f_{0,6})$ axis − $({}_{\mathrm{D}}f_{0,6})$ rim	+ 34·80	+ 22·59	+ 11·83
$({}_{\mathrm{D}}a_6')$ rim − $({}_{\mathrm{V}}a_6')$ rim	− 9·17	− 5·17	− 2·01
	A. Case 1*a*.	A. Case 2*a*.	A. Case 4*a*.
			

PRECEDING

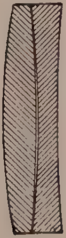
FIRST SURFACE OF THE BACK LENS

$h_0 = 30$ mm.

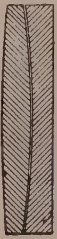
${}_{D}n' = 1{\cdot}613\,58$; ${}_{V}n' = 1{\cdot}642\,52$

Case 5*a*.	Case 11*a*.	Case 18*a*.	Case 19*a*.
+ 174·39	+ 420·00	− 976·63	− 226·92
8·0	8·0	0·8	8·0
− 572·36	− 200·33	− 119·15	− 83·956
0·39	0·01	0·0965	0·324
− 385·38	− 205·52	− 116·81	− 79·719
5·0	5·0	5·0	5·0
+ 659·29	− 1 101·1	− 218·72	− 120·51
702·52	713·43	724·95	736·26
702·52	713·43	724·96	736·26
702·53	713·43	724·95	736·27
703·21	713·89	726·14	739·07
720·00	720·00	720·00	720·00
720·26	720·00	719·74	719·48
716·07	719·92	722·07	723·65
716·90	720·32	722·85	725·74
2° 24′ 4″	2° 23′ 18″	2° 22′ 52″	2° 22′ 34″
+ 6 43 43	+ 6 40 18	+ 6 56 58	+ 7 40 57
− 4 19 39	− 4 17 0	− 4 34 6	− 5 18 23
+ 3 23 57	+ 1 23 57	− 0 36 3	− 2 36 3
+ 3 19 46	+ 5 16 21	+ 7 33 1	+ 10 17 0
− 4 16 4	− 5 46 55	− 8 30 59	− 11 55 25
− 0 3 35	+ 1 29 55	+ 3 56 53	+ 6 37 2
0·00	0·00	− 0·01	0·00
− 0·01	0·00	0·00	− 0·01
− 0·26	0·00	+ 0·26	+ 0·52
+ 3·93	+ 0·08	− 2·07	− 3·65
− 0·68	− 0·46	− 1·19	− 2·80

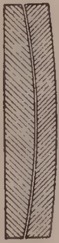
A. Case 5*a*.



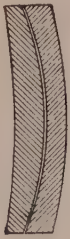
A. Case 11*a*.



A. Case 18*a*.



A. Case 19*a*.



Commencing with the same crown lenses of the objectives in Table A (*a*), there has been computed the set of objectives in Table A (*b*), in which, however, the distribution of the refractions at the surfaces of the flint lens is arranged in the opposite manner to the first table. It will be seen that for each crown lens there may be calculated a second flint lens to form a combination of the same true focal length, in which the spherical and chromatic aberrations are removed, but that, in order to eliminate the chromatic aberration, the distribution of the refractions in both lenses must be changed so that only the first radius of the crown lens can be retained. It is necessary so to alter the second radius that for the same true focal length achromatism will be again attained. If the radii of the lens surfaces are compared it will be seen that, proceeding from Case 1 *b* to Case 5 *b*, the lenses become flatter, and thereafter again more curved. As has been already explained in connection with the similar alteration of form in the first table, this arises from the fact that the spherical aberration of the crown lens first diminishes and then increases. From Case 1 *b* to Case 11 *b* the difference between the true focal length and the back focal length diminishes from 55·77 mm. to 27·16 mm. Thereafter it increases again to 34·16 mm. Zero value, however, is never reached, as the back focal length always remains smaller than the true focal length. In the first case (1 *b*) the total deviation reaches its greatest value, 2° 22′ 39″, and thereafter diminishes throughout the entire table to a value of 2° 3′ 3″. The apparent increase indicated in Case 2 *b* is due to uncertainty in the computation arising from the large values of the angles that occur. The refraction of the individual lenses is also very great in Case 1 *b* and diminishes up to Case 5 *b*. Thereafter, proceeding towards the right, it again increases.

As regards the Gauss-condition error, it will be seen that the signs are in all cases opposite to those of the preceding table. It decreases from the value +0·97 mm. in Case 1 *b* to a minimum value of +0·89 mm. in Case 2 *b*. Thereafter it increases to a value +62·81 mm. in Case 19 *b*. The errors of the two colours at the principal points have the same sign as those indicated in the preceding table. They decrease from left to right up to Case 11 *b* from the value −1·23 mm. to −0·56 mm. Thereafter they increase again to the value −0·88 mm. in

Case 19 *b*. The error against the sine condition in Case 1 *b* is of opposite sign to that of Case 1 *a*, and, proceeding towards Case 2 *b*, this error decreases from −3·22 mm. to −2·87 mm., whence it again increases without change of sign to a value of −118·27 mm. in Case 19*b*. From these figures it will be observed that the values of this error in the first case of the Table A (*b*)—that is, before the minimum is reached—are much smaller than those indicated in the first table, and that in the later cases they are much greater. Further, the last cases exhibit particularly large errors, for the reason that the alteration in the distance between the lenses from the axis to the rim is much greater.

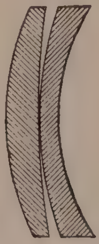
(*b*) THE STRONGER REFRACTION AT THE

$f_{0,6} = 720$ mm.;

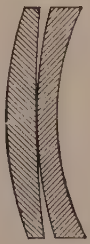
$_{\mathrm{D}}n = 1·518\,06$; $_{\mathrm{V}}n = 1·533\,56$;

Particulars.	Case 1*b*.	Case 2*b*.	Case 4*b*.
r_0	+ 67·282	+ 83·159	+ 111·50
d_1	8·0	8·0	8·0
r_2	+ 172·36	+ 286·95	+ 1 179·6
d_3	0·01	0·01	0·01
r_4	+ 83·245	+ 105·71	+ 140·70
d_5	5·0	5·0	5·0
r_6	+ 53·917	+ 65·937	+ 82·231
$({}_{\mathrm{D}}a_6')$ axis	664·23	673·81	683·43
$({}_{\mathrm{V}}a_6')$ axis	664·22	673·81	683·43
$({}_{\mathrm{D}}a_6')$ rim	664·20	673·82	683·44
$({}_{\mathrm{V}}a_6')$ rim	663·23	672·93	682·22
$({}_{\mathrm{D}}f_{0,6})$ axis	720·00	720·00	720·00
$({}_{\mathrm{V}}f_{0,6})$ axis	721·23	721·01	720·74
$({}_{\mathrm{D}}f_{0,6})$ rim	723·22	722·87	724·24
$({}_{\mathrm{V}}f_{0,6})$ rim	723·60	723·10	723·94
$({}_{\mathrm{D}}\beta_{0,6})$ rim	$2°\ 22'\ 39''$	$2°\ 22'\ 43''$	$2°\ 22'\ 26''$
$\delta_0 + \delta_2$	+ 9 10 34	+ 8 10 36	+ 7 27 28
$\delta_4 + \delta_6$	− 6 47 55	− 5 47 53	− 5 5 2
δ_0	+ 9 23 57	+ 7 23 57	+ 5 23 57
δ_2	− 0 13 23	+ 0 46 39	+ 2 3 31
δ_4	+ 4 17 17	+ 2 58 50	+ 1 43 38
δ_6	− 11 5 12	− 8 46 43	− 6 48 40
$({}_{\mathrm{D}}a_6')$ axis − $({}_{\mathrm{V}}a_6')$ axis	+ 0·01	0·00	0·00
$({}_{\mathrm{D}}a_6')$ axis − $({}_{\mathrm{D}}a_6')$ rim	+ 0·03	− 0·01	− 0·01
$({}_{\mathrm{D}}f_{0,6})$ axis − $({}_{\mathrm{V}}f_{0,6})$ axis	− 1·23	− 1·01	− 0·74
$({}_{\mathrm{D}}f_{0,6})$ axis − $({}_{\mathrm{D}}f_{0,6})$ rim	− 3·22	− 2·87	− 4·24
$({}_{\mathrm{D}}a_6')$ rim − $({}_{\mathrm{V}}a_6')$ rim	+ 0·97	+ 0·89	+ 1·22

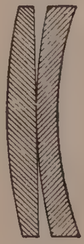
A. Case 1*b*.



A. Case 2*b*.



A. Case 4*b*.



PRECEDING

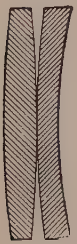
SECOND SURFACE OF THE BACK LENS

$h_0 = 30$ mm.

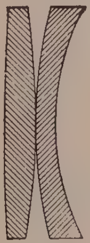
${}_Dn' = 1{\cdot}613\,58$; ${}_Vn' = 1{\cdot}642\,52$

Case 5*b*.	Case 11*b*.	Case 18*b*.	Case 19*b*.
+ 174·39	+ 420·00	− 976·63	− 226·92
8·0	8·0	8·0	8·0
− 448·09	− 172·00	− 100·95	− 66·943
0·01	0·01	0·01	0·01
+ 150·77	+ 104·44	+ 69·048	+ 45·564
5·0	5·0	5·0	5·0
+ 87·501	+ 68·073	+ 48·573	+ 32·858
690·59	692·84	691·81	685·83
690·58	692·81	691·82	685·81
690·59	692·83	691·80	685·87
688·22	687·42	677·56	623·06
720·00	720·00	720·01	719·99
720·59	720·56	720·70	720·87
730·04	744·03	770·20	838·27
728·53	739·61	756·55	764·48
2° 21′ 19″	2° 18′ 39°	2° 13′ 52″	2° 3′ 3″
+ 7 10 11	+ 7 27 53	+ 8 34 11	+ 11 38 30
− 4 48 52	− 5 9 14	− 6 20 19	− 9 35 27
+ 3 23 57	+ 1 23 57	− 0 36 3	− 2 36 3
+ 3 46 14	+ 6 3 56	+ 9 10 14	+ 14 14 33
+ 1 31 45	+ 3 18 25	+ 6 3 45	+ 9 35 29
− 6 20 37	− 8 27 39	− 12 24 4	− 19 10 56
+ 0·01	+ 0·03	− 0·01	+ 0·02
0·00	+ 0·01	+ 0·01	− 0·04
− 0·59	− 0·56	− 0·69	− 0·88
− 10·04	− 24·03	− 50·19	− 118·28
+ 2·37	+ 5·41	+ 14·24	+ 62·81

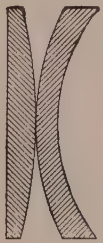
A. Case 5*b*.



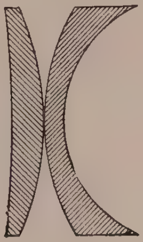
A. Case 11*b*.



A. Case 18*b*.



A. Case 19*b*.



Tables B (*a*) and B (*b*) contain those objectives in which the flint lens is in front. In Table B (*a*) the refractions at the first surfaces are again so chosen that the values of δ_0 differ by steps of 2°. Thus, in Case 1 *a*, the deviation $\delta_0 = +7° \, 38' \, 40''$; in Case 2 *a*, $+5° \, 38' \, 40''$; in Case 3 *a*, $+3° \, 38' \, 40''$; and in Case 5 *a*, $+1° \, 38' \, 40''$. The cases in which $\delta_0 = -0° \, 21' \, 20''$, $-2° \, 21' \, 20''$, and $-4° \, 21' \, 20''$, must be omitted, since in these cases the spherical aberration of the flint lens is so small that it is no longer possible for it to compensate the error of the equally refracting crown lens. In the last case (11 *a*) the deviation $\delta_0 = -6° \, 21' \, 20''$.

At the back lens of the objective, the stronger refraction, which is *towards* the axis, takes place at its first surface, while the refraction *from* the axis, or the weaker refraction *towards* the axis, takes place at the second surface. It will be observed that the flint lens is a meniscus of negative focal length, and that from Case 1 *a* to 5 *a* the first radius becomes progressively flatter, being concave in Case 11 *a*. From Case 1 *a* to 3 *a* the crown lens is a meniscus of positive focal length. In Cases 5 *a* and 11 *a* it is biconvex, the first convex surface becoming progressively flatter.

As regards the total refraction, the various objectives show only small differences. Between Cases 5 *a* and 11 *a* the value reaches a minimum which differs from that of Case 5 *a* only by about one second. As regards the refraction of the individual lenses, it should be noted that the negative deviation of the flint lens reaches its greatest value in the first example and a minimum between Cases 5 *a* and 11 *a*. Thereafter the value again increases. Also, the positive deviation of the crown lens approaches a minimum between Cases 5 *a* and 11 *a*, and thereafter increases.

Similarly the back focal lengths differ less from the true focal lengths. In Case 1 *a* the back focal length is 33·90 mm. shorter than the focal length. Up to Case 5 *a* it remains shorter, and thereafter increases until in Case 11 *a* it is 12·10 mm. greater than the focal length.

The errors in the fulfilment of the Gauss condition are in the first instance such that the violet ray has a longer back focal length by an amount 3·20 mm. It becomes gradually smaller, until in Case 5 *a* it is 0·36 mm. Reversal of the sign takes place at Case 11*a*, where the greater value 2·85 mm. is reached.

The error of chromatic difference of magnification varies in a similar manner, being zero in Case 5*a* and −0·64 mm. and +0·53 mm. in Cases 1*a* and 11*a* respectively. Between Cases 3*a* and 5*a* the sine condition error passes through zero, and a change of sign takes place again between Cases 5*a* and 11*a*. Of the objectives contained in this table, 5*a* has the smallest errors, the greatest being the sine condition error, which, however, only amounts to −0·63 mm.

In Table B (*b*) the first radius is again chosen, as in the previous Tables B (*a*), but the refractions at the crown-glass lens are reversed. Thus the objectives comprise oppositely-placed menisci, with the exception of the last case (11*b*), in which the crown lens is bi-convex. Between Cases 5*b* and 11*b*, and in the neighbourhood of the latter, there must be an example in which all the errors are very small. Arrangements consisting of oppositely placed menisci show very large departures from the sine and Gauss conditions, both of which change their sign between Cases 5*b* and 11*b*, while the errors of chromatic magnification are all of the same sense and not very large. The latter first decrease in value until a minimum is reached between 5*b* and 11*b*, and thereafter they increase.

B. FLINT

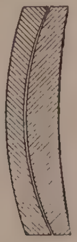
(a) THE STRONGER REFRACTION AT THE

$f_{0,6} = 720 \cdot 0$ mm.;

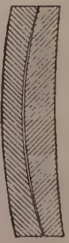
${}_{D}n = 1 \cdot 518\,06$; ${}_{V}n = 1 \cdot 533\,56$;

Particulars.	Case 1*a*.	Case 2*a*.
r_0	+ 88·961	+ 118·37
d_1	5·0	5·0
r_2	+ 62·120	+ 77·564
d_3	0·35	0·2
r_4	+ 64·809	+ 79·915
d_5	8·0	8·0
r_6	+ 134·42	+ 208·82
$({}_{D}a_6')$ axis	686·10	694·46
$({}_{V}a_6')$ axis	686·10	694·48
$({}_{D}a_6')$ rim	686·12	694·51
$({}_{V}a_6')$ rim	689·32	696·38
$({}_{D}f_{0,6})$ axis	720·00	720·00
$({}_{V}f_{0,6})$ axis	720·64	720·47
$({}_{D}f_{0,6})$ rim	716·23	717·34
$({}_{V}f_{0,6})$ rim	720·31	719·82
$({}_{D}\beta_{0,6})$ rim	2° 24′ 2″	2° 23′ 49″
$\delta_0 + \delta_2$	− 6 3 15	− 5 10 49
$\delta_4 + \delta_6$	+ 8 27 17	+ 7 34 38
δ_0	+ 7 38 41	+ 5 38 40
δ_2	− 13 41 56	− 10 49 29
δ_4	+ 11 51 5	+ 9 29 8
δ_6	− 3 23 48	− 1 54 30
$({}_{D}a_6')$ axis − $({}_{V}a_6')$ axis	0·00	− 0·02
$({}_{D}a_6')$ axis − $({}_{D}a_6')$ rim	− 0·02	− 0·05
$({}_{D}f_{0,6})$ axis − $({}_{V}f_{0,6})$ axis	− 0·64	− 0·47
$({}_{D}f_{0,6})$ axis − $({}_{D}f_{0,6})$ rim	+ 3·77	+ 2·66
$({}_{D}a_6')$ rim − $({}_{V}a_6')$ rim	− 3·20	− 1·87

B. Case 1*a*.



B. Case 2*a*.



PRECEDING

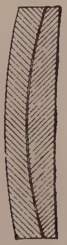
FIRST SURFACE OF THE BACK LENS

h_0 = 30·0 mm.

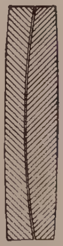
$_{D}n'$ = 1·613 58; $_{V}n'$ = 1·642 52

Case 3*a*.	Case 4*a*.	Case 5*a*.	Case 11*a*.
+ 181·02	+ 308·20	+ 398·25	− 105·73
5·0	5·0	5·0	5·0
+ 103·55	+ 138·18	+ 154·57	− 193·78
0·04	0·01	0·01	0·01
+ 104·11	+ 136·13	+ 152·38	+ 187·84
8·0	8·0	8·0	8·0
+ 476·39	− 17 654·1	− 1 235·7	− 453·67
703·21	710·42	712·88	732·10
703·19	710·41	712·86	732·07
703·22	710·43	712·91	732·06
704·10	710·87	713·27	729·21
720·00	720·00	720·00	720·00
720·22	720·07	720·00	719·47
718·66	720·00	720·63	706·47
719·90	720·58	721·06	702·75
2° 23′ 33″	2° 23′ 17″	2° 23′ 9″	2° 26′ 2″
− 4 36 5	− 4 20 41	− 4 17 1	− 4 38 50
+ 6 59 38	+ 6 43 58	+ 6 40 10	+ 7 4 52
+ 3 38 40	+ 2 7 39	+ 1 38 40	− 6 21 21
− 8 14 45	− 6 28 20	− 5 55 41	+ 1 42 31
+ 7 22 57	+ 5 53 5	+ 5 23 2	+ 4 54 5
− 0 23 19	+ 0 50 53	+ 1 17 8	+ 2 10 47
+ 0·02	+ 0·01	+ 0·02	+ 0·03
− 0·01	− 0·01	− 0·03	+ 0·04
− 0·22	− 0·07	0·00	+ 0·53
+ 1·34	0·00	− 0·63	+ 13·53
− 0·88	− 0·44	− 0·36	+ 2·85

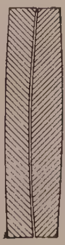
B. Case 3*a*.



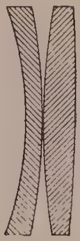
B. Case 4*a*.



B. Case 5*a*.



B. Case 11*a*.



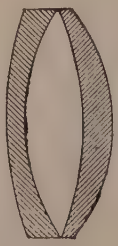
(*b*) THE STRONGER REFRACTION AT THE

$f_{0,6} = 720$ mm.;

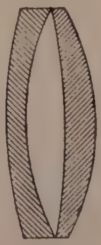
${}_{\mathrm{D}}n = 1{\cdot}518\,06$; ${}_{\mathrm{V}}n = 1{\cdot}533\,56$;

Particulars.	Case 1*b*.	Case 2*b*.
r_0	+ 88·961	+ 118·37
d_1	5·0	5·0
r_2	+ 54·143	+ 71·193
d_3	13·93	9·85
r_4	− 77·827	− 122·96
d_5	8·0	8·0
r_6	− 45·457	− 61·025
$({}_{\mathrm{D}}a_6')$ axis	788·77	760·57
$({}_{\mathrm{V}}a_6')$ axis	788·73	760·57
$({}_{\mathrm{D}}a_6')$ rim	788·77	760·55
$({}_{\mathrm{V}}a_6')$ rim	854·81	781·71
$({}_{\mathrm{D}}f_{0,6})$ axis	720·00	720·00
$({}_{\mathrm{V}}f_{0,6})$ axis	717·61	718·57
$({}_{\mathrm{D}}f_{0,6})$ rim	811·98	772·32
$({}_{\mathrm{V}}f_{0,6})$ rim	879·15	793·62
$({}_{\mathrm{D}}\beta_{0,6})$ rim	2° 7′ 23″	2° 13′ 34″
$\delta_0 + \delta_2$	− 9 58 37	− 6 45 9
$\delta_4 + \delta_6$	+ 12 5 40	+ 8 58 43
δ_0	+ 7 38 41	+ 5 38 40
δ_2	− 17 37 18	− 12 23 49
δ_4	− 4 4 2	− 2 24 5
δ_6	+ 16 9 42	+ 11 22 48
$({}_{\mathrm{D}}a_6')$ axis − $({}_{\mathrm{V}}a_6')$ axis	+ 0·04	0·00
$({}_{\mathrm{D}}a_6')$ axis − $({}_{\mathrm{D}}a_6')$ rim	0·00	+ 0·02
$({}_{\mathrm{D}}f_{0,6})$ axis − $({}_{\mathrm{V}}f_{0,6})$ axis	+ 2·39	+ 1·43
$({}_{\mathrm{D}}f_{0,6})$ axis − $({}_{\mathrm{D}}f_{0,6})$ rim	− 91·98	− 52·32
$({}_{\mathrm{D}}a_6')$ rim − $({}_{\mathrm{V}}a_6')$ rim	− 66·04	− 21·16

B. Case 1*b*.



B. Case 2*b*.



PRECEDING

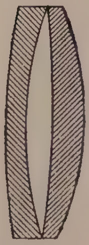
SECOND SURFACE OF THE BACK LENS

$h_0 = 30$ mm.

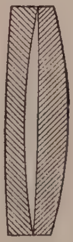
${}_Dn' = 1{\cdot}613\,58$; $n' = 1{\cdot}642\,52$

Case 3*b*.	Case 5*b*.	Case 11*b*.
+ 181·02	+ 398·25	− 105·73
5·0	5·0	5·0
+ 97·299	+ 147·75	− 198·29
6·57	3·53	0·01
− 223·59	− 964·33	+ 694·21
8·0	8·0	8·0
− 81·592	− 116·99	− 143·39
745·61	735·03	741·83
745·59	735·05	741·87
745·60	735·03	741·83
754·67	738·82	741·03
720·01	720·00	720·00
719·05	719·42	719·36
751·80	737·29	717·09
760·85	740·92	715·42
2° 17′ 13″	2° 19′ 55″	2° 23′ 52″
− 5 21 24	− 4 37 20	− 4 46 32
+ 7 38 37	+ 6 57 15	+ 7 10 24
+ 3 38 40	+ 1 38 40	− 6 21 21
− 9 0 4	− 6 16 0	+ 1 34 49
− 0 45 50	− 0 58 24	+ 2 0 4
+ 8 24 27	+ 5 58 51	+ 5 10 20
+ 0·02	− 0·02	− 0·04
+ 0·01	0·00	0·00
+ 0·96	+ 0·58	+ 0·64
− 31·79	− 17·29	+ 2·91
− 9·07	− 3·79	+ 0·80

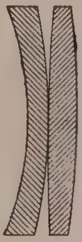
B. Case 3*b*.



B. Case 5*b*.



B. Case 11*b*.



Some Special Cases of Doublet Objectives corrected for Spherical Aberration

In addition to the objectives indicated in the four preceding tables, there has been calculated a series of special cases which satisfy particular conditions. They have been grouped in two tables in such a way that the relative position in the previous tables of any particular example can at once be seen.

Table A (*a*), pp. 134 to 137, contains the following examples. The first case 3 *a* has been so calculated that the last surface of the objective is plane, while the first radius r_0 is more curved than r_2. The departures from the sine and Gauss conditions are considerable, being +16·31 mm. and −3·44 mm. respectively. The whole series of special cases 6 *a* to 10 *a* lies between Cases 5 *a* and 11 *a* of Table A (*a*), p. 119. The conditions which these examples satisfy are as follows.

In Case 6 *a* the last surface has its centre at the focal point. In this example also the sine condition error is considerable. Case 7 *a* represents a cemented objective in which the first radius r_0 is more curved than r_2. The errors of this objective are smaller than those of the preceding example. Still smaller are the errors of example 8 *a*, in which the crown lens is equi-convex. There has been inserted at this position of the table an additional case 8 *a*,* in which the separation of the lenses is 10 mm., in order to show how the chromatic difference of magnification and the sine-condition error vary while the Gauss-condition error is improved. It will be seen that as the result of this alteration the error in the Gauss condition has been improved from a value −0·31 mm. to −0·17 mm., while the chromatic difference of magnification has increased from −0·07 mm. to −0·98 mm., and the sine-condition error has decreased from +1·28 mm. to +0·67 mm. In order to satisfy the Gauss condition the separation of the lenses must be increased still further, in which case the chromatic difference of magnification will be about −2·0 mm. and the sine-condition error about zero. There will then be obtained an objective which will be bad as regards chromatic difference of magnification, but good as regards the sine and Gauss conditions. These constructions, upon which the larger

English and American objectives are based,[‡] cannot be recommended, as there are various better forms available.

In Case 9 *a* the last surface is flat, but the radius r_2 is more curved than r_0, which is the reverse of Case 3 *a*. Compared with Case 8 *a*, the departure from the Gauss condition is a little greater, but the other errors are in general somewhat smaller. Case 10 *a* completely satisfies the Gauss condition. This result is attained by the separation of the lenses from one another, which, however, makes the sine-condition error and the chromatic difference of magnification greater than in Case 9 *a*.

Between the objectives 11*a* and 18*a* of Table A (*a*), p. 119, a series of six special examples has been calculated. In Case 12*a*, for which, in addition to the chromatic and spherical aberrations, the chromatic difference of magnification and the sine-condition error have been completely removed, there remains only a small error as regards the Gauss condition. This objective is the best of those in Tables A (*a*) when the angular field is large. In Case 13 *a* the principal points lie in one plane and not upon a sphere having the focal point as its centre, as in the previous case. In Case 14 *a* the spherical aberration has been removed in both directions. Both the last-mentioned examples show in comparison with the best objective, Case 12 *a*, an increase of the chromatic difference of magnification, and as regards the departures from the sine and Gauss conditions.

The next objective 15 *a* has been so computed that the back focal lengths are equal to the true focal lengths. The image errors are greater than those of Case 14 *a*. A second cemented objective is indicated in Case 16 *a*, in which the first radius r_0 is flatter than r_2, which is the reverse of Case 7 *a*. This second cemented objective exhibits errors of chromatic difference of magnification, as well as departures from the sine and Gauss conditions greater than those of Case 7 *a*. Finally, in Case 17 *a*, the first surface has been chosen plane. The errors in this example are all greater in comparison with those of Case 15 *a*.

As an addition to Table A (*b*), p. 122, there has been computed only one special example in which, by a reduction of the thickness of the flint lens of the objective in Case 2 *b*, the departure from the Gauss condition has been reduced to zero. This lens is indicated in the last column 2 *b**, p. 137. It will be observed that in comparison with Case 2*b* the objective is considerably better as regards chromatic difference of magnification, namely −0·57 mm. as compared with −1·01 mm., and as regards the sine-condition error, namely −1·07 mm. as compared with −2·87 mm. This objective corresponds in form with those commonly known as Gauss objectives. In practice the thicknesses of the flint and crown lenses must be somewhat differently proportioned in order to obtain a greater thickness of the flint lens at its centre.

[‡] It should be remembered that this statement was written in 1891. It is not necessarily applicable at the present time.—ED.

Between Cases 5*a* and 11*a* of Table B (*a*), p. 127, there have been computed the following special examples. In Case 6*a*, p. 128, the spherical aberration has been eliminated in both directions. When this objective is compared with Case 14*a* of Table A (*a*), p. 136, in which the crown glass is in front, it will be seen that in the former case B, 6*a*, having the flint in front, the fulfilment of the Gauss condition is decidedly better than in the latter case, the error in the one being about half that of the other, namely −0·30 mm. as compared with −0·68 mm. At the same time, the chromatic differences of magnification and the sine-condition errors are in both cases equally great, allowing for inaccuracies of calculation. In Case 6*a** the Gauss condition has been satisfied by increasing the separation.

Cases A, 8*a** and B, 6*a** have been evolved from A, 8*a* and B, 6*a* respectively by increasing the lens separations, and the effect upon the Gauss and sine conditions and the chromatic difference of magnification have also been determined. It will be seen that, while the departures from the Gauss and sine conditions can be made simultaneously zero, the chromatic difference of magnification has the greatest value, and particularly in the combination having the crown in front. Example B, 7*a* is a cemented objective exhibiting a greater departure from the sine condition, which increases still further in the succeeding special examples. In Case B, 8*a* the crown lens is equi-convex. Examples 9*a* and 10*a* constitute the limits of the possible cases in which the flint is in front. Of these 9*a* has the weakest possible refraction at r_0 towards the axis, and 10*a* the weakest refraction at r_0 from the axis.

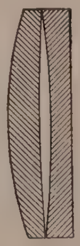
A. SPECIAL CASES,

(*a*) THE STRONGER REFRACTION AT THE

$f_{0,6} = 720$ mm.; $h_0 = 30$ mm.; ${}_{D}n = 1{\cdot}518\,06$;

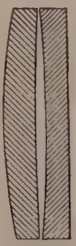
Particulars.	Case 3*a*.	Case 6*a*.	Case 7*a*.
r_0	+ 96·822	+ 183·33	+ 266·92
d_1	8·0	8·0	8·0
r_2	+ 392·49	− 495·82	− 273·91
d_3	3·13	0·30	0·0
r_4	− 219·65	− 374·04	− 273·91
d_5	5·0	5·0	5·0
r_6	∞	+ 703·45	+ 2 920·7
$({}_{D}a_6')$ axis	687·43	703·45	709·07
$({}_{V}a_6')$ axis	687·42	703·45	709·07
$({}_{D}a_6')$ rim	687·44	703·45	709·10
$({}_{V}a_6')$ rim	690·88	704·05	709·50
$({}_{D}f_{0,6})$ axis	720·00	720·00	720·00
$({}_{V}f_{0,6})$ axis	720·63	720·24	720·10
$({}_{D}f_{0,6})$ rim	703·69	716·52	718·72
$({}_{V}f_{0,6})$ rim	707·32	717·02	719·15
$({}_{D}\beta_{0,6})$ rim	2° 26′ 36″	2° 23′ 59″	2° 23′ 32″
$\delta_0 + \delta_2$	+7 17 19	+6 42 54	+6 39 3
$\delta_4 + \delta_6$	−4 50 43	−4 18 55	−4 15 31
δ_0	+6 16 22	+3 13 48	+2 12 27
δ_2	+1 0 57	+3 29 6	+4 26 36
δ_4	−5 46 28	−4 18 55	−4 56 51
δ_6	+0 55 45	0 0 0	+0 41 20
$({}_{D}a_6')$ axis − $({}_{V}a_6')$ axis	+ 0·01	0·00	0·00
$({}_{D}a_6')$ axis − $({}_{D}a_6')$ rim	− 0·01	0·00	− 0·03
$({}_{D}f_{0,6})$ axis − $({}_{V}f_{0,6})$ axis	− 0·63	− 0·24	− 0·10
$({}_{D}f_{0,6})$ axis − $({}_{D}f_{0,6})$ rim	+ 16·31	+ 3·48	+ 1·28
$({}_{D}a_6')$ rim − $({}_{V}a_6')$ rim	− 3·44	− 0·60	− 0·40

A. Case 3*a*.



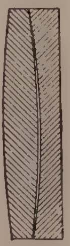
Last surface plane, r_0 more curved than r_2.

A. Case 6*a*.



Last radius described about the focal point as centre.

A. Case 7*a*.

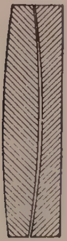
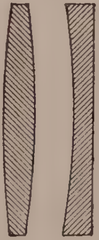
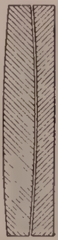
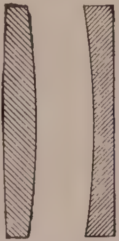


Cemented objective, r_0 more curved than r_2.

CROWN PRECEDING

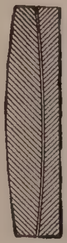
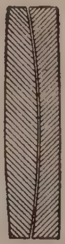
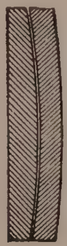
FIRST SURFACE OF THE BACK LENS

$_Vn = 1{\cdot}533\,56$; $_Dn' = 1{\cdot}613\,58$; $_Vn' = 1{\cdot}642\,52$

Case 8*a*.	Case 8*a*.*	Case 9*a*.	Case 10*a*.
+ 270·64	+ 270·64	+ 298·93	+ 364·65
8·0	8·0	8·0	8·0
− 270·64	− 231·64	− 247·47	− 174·27
0·01	10·0	0·01	15·34
− 271·51	− 218·19	− 251·38	− 159·22
5·0	5·0	5·0	5·0
+ 3 297·4	+ 8 075·9	∞	− 696·41
709·20	679·75	710·30	664·54
709·17	679·71	710·29	664·55
709·20	679·75	710·31	664·56
709·51	679·92	710·71	664·56
720·00	720·00	720·00	720·00
720·07	720·98	720·06	721·59
718·72	719·33	719·10	721·34
719·09	720·56	719·50	722·98
2° 23′ 32″	2° 23′ 25″	2° 23′ 28″	2° 23′ 1″
+ 6 38 43	+ 7 13 44	+ 6 38 58	+ 7 43 17
− 4 15 11	− 4 50 19	− 4 15 30	− 5 20 16
+ 2 10 37	+ 2 10 37	+ 1 58 9	+ 1 36 45
+ 4 28 6	+ 5 3 7	+ 4 40 49	+ 6 6 32
− 4 58 2	− 5 40 16	− 5 10 5	− 6 52 3
+ 0 42 51	+ 0 49 57	+ 0 54 35	+ 1 31 47
+ 0·03	+ 0·04	+ 0·01	− 0·01
0·00	0·00	− 0·01	− 0·02
− 0·07	− 0·98	− 0·06	− 1·59
+ 1·28	+ 0·67	+ 0·90	− 1·34
− 0·31	− 0·17	− 0·40	0·00
A. Case 8*a*.	A. Case 8*a*.*	A. Case 9*a*.	A. Case 10*a*.
			
Crown lens equi-convex.	English and American type of large objective.	The last surface plane, r_2 more curved than r_0.	The Gauss condition satisfied.

(*a*) THE STRONGER REFRACTION AT THE

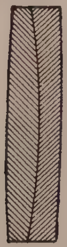
$f_{0,6} = 720$ mm.; $h_0 = 30$ mm.; ${}_{D}n = 1 \cdot 51806$;

Particulars.	Case 12*a*.	Case 13*a*.	Case 14*a*.
r_0	+ 441·45	+ 708·76	+ 919·62
d_1	8·0	8·0	8·0
r_2	− 196·12	− 168·25	− 159·52
d_3	0·01	0·01	0·01
r_4	− 201·12	− 171·13	− 161·47
d_5	5·0	5·0	5·0
r_6	− 977·97	− 525·54	− 444·56
$({}_{D}a_6')$ axis	713·81	716·69	717·81
$({}_{V}a_6')$ axis	713·81	716·69	717·83
$({}_{D}a_6')$ rim	713·81	716·67	717·81
$({}_{V}a_6')$ rim	714·26	717·23	718·49
$({}_{D}f_{0,6})$ axis	720·00	720·00	720·00
$({}_{V}f_{0,6})$ axis	719·99	719·93	719·93
$({}_{D}f_{0,6})$ rim	720·01	720·60	720·83
$({}_{V}f_{0,6})$ rim	720·39	721·03	721·32
$({}_{D}\beta_{0,6})$ rim	2° 23′ 17″	2° 23′ 10″	2° 23′ 7″
$\delta_0 + \delta_2$	+ 6 40 8	+ 6 42 29	+ 6 43 55
$\delta_4 + \delta_6$	− 4 16 51	− 4 19 19	− 4 20 48
δ_0	+ 1 19 51	+ 0 49 41	+ 0 38 17
δ_2	+ 5 20 17	+ 5 52 48	+ 6 5 38
δ_4	− 5 51 14	− 6 28 22	− 6 43 37
δ_6	+ 1 34 23	+ 2 9 3	+ 2 22 49
$({}_{D}a_6')$ axis − $({}_{V}a_6')$ axis	0·00	0·00	− 0·02
$({}_{D}a_6')$ axis − $({}_{D}a_6')$ rim	0·00	+ 0·02	0·00
$({}_{D}f_{0,6})$ axis − $({}_{V}f_{0,6})$ axis	+ 0·01	+ 0·07	+ 0·07
$({}_{D}f_{0,6})$ axis − $({}_{D}f_{0,6})$ rim	− 0·01	− 0·60	− 0·83
$({}_{D}a_6')$ rim − $({}_{V}a_6')$ rim	− 0·45	− 0·56	− 0·68
	A. Case 12*a*.	A. Case 13*a*.	A. Case 14*a*.
			
	Sine condition satisfied.	The principal points in one plane.	Spherical aberration in both directions eliminated.

CROWN PRECEDING

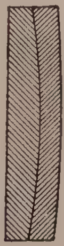
FIRST SURFACE OF THE BACK LENS $_{\mathrm{v}}n = 1{\cdot}533\,56$; $_{\mathrm{D}}n' = 1{\cdot}613\,58$; $_{\mathrm{v}}n' = 1{\cdot}642\,52$			(*b*) The Stronger Refraction at the Second Surface of the Back Lens.
Case 15*a*.	Case 16*a*.	Case 17*a*.	Case 2*b*.*
+ 2 192·0	+ 3 519·9	∞	+ 83·159
8·0	8·0	8·0	8·0
− 145·11	− 141·36	− 136·00	+ 234·218
0·01	0·0	0·045 9	0·01
− 145·46	− 141·36	− 135·35	+ 100·096
5·0	5·0	5·0	0·5
− 340·94	− 320·37	− 291·11	+ 68·540
719·99	720·62	721·49	694·10
719·98	720·66	721·49	694·14
720·00	720·62	721·48	694·11
720·77	721·52	722·37	694·12
720·00	720·00	720·00	720·00
719·84	719·88	719·81	720·57
721·22	721·31	721·50	721·07
721·76	721·94	722·12	721·81
2° 23′ 2″	2° 23′ 1″	2° 22′ 59″	2° 23′ 4″
+ 6 46 26	+ 6 48 8	+ 6 49 15	+ 7 29 26
− 4 23 24	− 4 25 7	− 4 26 16	− 5 6 22
+ 0 16 3	+ 0 10 0	0 0 0	+ 7 23 57
+ 6 30 23	+ 6 38 8	+ 6 49 15	+ 0 5 29
− 7 13 34	− 7 22 52	− 7 36 40	+ 3 35 58
+ 2 50 10	+ 2 57 45	+ 3 10 24	− 8 42 20
+ 0·01	− 0·04	0·00	− 0·04
− 0·01	0·00	+ 0·01	− 0·01
+ 0·16	+ 0·12	+ 0·19	− 0·57
− 1·22	− 1·31	− 1·50	− 1·07
− 0·77	− 0·90	− 0·89	− 0·01

A. Case 15*a*.



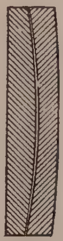
Back focal length equal to the true focal length.

A. Case 16*a*.



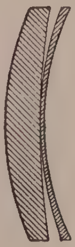
Cemented objective, r_0 flatter than r_2.

A. Case 17*a*.



First surface plane.

A. Case 2*b*.*



The Gauss condition satisfied.

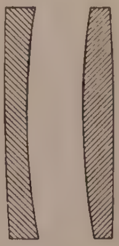
B. SPECIAL CASES,

(*a*) THE STRONGER REFRACTION AT THE

$f_{0,6} = 720$ mm.; $h_0 = 30$ mm.; ${}_{D}n = 1 \cdot 518\,06$;

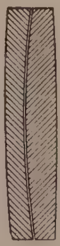
Particulars.	Case 6*a*.*	Case 6*a*.
r_0	+ 444·56	+ 444·56
d_1	5·0	5·0
r_2	+ 149·04	+ 161·48
d_3	15·0	0·01
r_4	+ 156·52	+ 159·52
d_5	8·0	8·0
r_6	− 790·02	− 919·62
$({}_{D}a_6')$ axis	743·59	713·79
$({}_{V}a_6')$ axis	743·61	713·72
$({}_{D}a_6')$ rim	743·63	713·76
$({}_{V}a_6')$ rim	743·67	714·06
$({}_{D}f_{0,6})$ axis	720·00	720·00
$({}_{V}f_{0,6})$ axis	718·68	719·92
$({}_{D}f_{0,6})$ rim	719·91	720·81
$({}_{V}f_{0,6})$ rim	718·58	721·17
$({}_{D}\beta_{0,6})$ rim	2° 23′ 18″	2° 23′ 7″
$\delta_0 + \delta_2$	− 4 50 41	− 4 15 34
$\delta_4 + \delta_6$	+ 7 13 59	+ 6 38 41
δ_0	+ 1 28 21	+ 1 28 21
δ_2	− 6 19 2	− 5 43 55
δ_4	+ 5 38 51	+ 5 11 48
δ_6	+ 1 35 8	+ 1 26 53
$({}_{D}a_6')$ axis − $({}_{V}a_6')$ axis	− 0·02	+ 0·07
$({}_{D}a_6')$ axis − $({}_{D}a_6')$ rim	− 0·04	+ 0·03
$({}_{D}f_{0,6})$ axis − $({}_{V}f_{0,6})$ axis	+ 1·32	+ 0·08
$({}_{D}f_{0,6})$ axis − $({}_{D}f_{0,6})$ rim	+ 0·09	− 0·81
$({}_{D}a_6')$ rim − $({}_{V}a_6')$ rim	− 0·04	− 0·30

B. Case 6*a*.*



The Gauss condition satisfied.

B. Case 6*a*.



Spherical aberration in both directions eliminated.

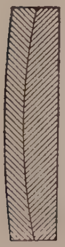
FLINT PRECEDING

FIRST SURFACE OF THE BACK LENS

${}_{\mathrm{V}}n = 1{\cdot}533\,56;\ {}_{\mathrm{D}}n' = 1{\cdot}613\,58;\ {}_{\mathrm{V}}n' = 1{\cdot}642\,52$

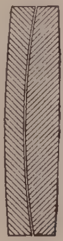
Case 7*a*.	Case 8*a*.	Case 9*a*.	Case 10*a*.
+ 575·24	+ 1 821·1	+ 2 287·7	− 119·82
5·0	5·0	5·0	5·0
+ 175·95	+ 222·08	+ 226·95	− 243·91
0·0	0·38	0·58	0·01
+ 175·95	+ 272·15	+ 318·43	+ 317·42
8·0	8·0	8·0	8·0
− 596·44	− 272·15	− 237·26	− 230·28
715·63	721·88	723·48	734·89
715·64	721·89	723·48	734·90
715·64	721·88	723·48	734·85
716·02	722·52	724·26	733·28
720·00	720·00	720·00	720·00
719·96	719·80	719·76	719·47
721·41	723·98	724·98	713·12
721·80	724·55	725·63	710·77
2° 23′ 0″	2° 22′ 30″	2° 22′ 18″	2° 24′ 40″
− 4 14 53	− 4 13 54	− 4 14 47	− 4 33 36
+ 6 37 53	+ 6 36 24	+ 6 37 5	+ 6 58 16
+ 1 8 14	+ 0 21 32	+ 0 17 9	− 5 34 24
− 5 23 7	− 4 35 26	− 4 31 56	+ 1 0 48
+ 4 50 15	+ 3 37 24	+ 3 18 33	+ 3 29 8
+ 1 47 38	+ 2 59 0	+ 3 18 32	+ 3 29 8
− 0·01	− 0·01	0·00	− 0·01
− 0·01	0·00	0·00	+ 0·04
+ 0·04	+ 0·20	+ 0·24	+ 0·53
− 1·41	− 3·98	− 4·98	+ 6·88
− 0·38	− 0·64	− 0·78	+ 1·57

B. Case 7*a*.



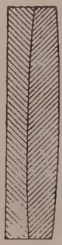
Cemented objective.

B. Case 8*a*.



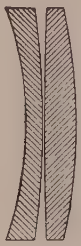
Crown lens equiconvex.

B. Case 9*a*.



Flattest possible flint lens, r_2 more curved than r_0.

B. Case 10*a*.



Flattest possible flint lens, r_2 flatter than r_0.

The objectives forming the subject of the preceding discussion have been computed for types of glass in common use. Which of the forms should be chosen in any particular case depends upon the conditions to be satisfied by the objective. When the power of the objective is small, in which case the limits of error may easily be within the power of appreciation of the eye, a cemented objective is to be preferred on account of the smaller loss of light, there being in such a case only two refractions between air and glass. In the case of objectives designed to cover a large angular field and in which measurements have to be made of the images formed by them, it is essential that a form should be chosen which, while completely satisfying the sine and chromatic difference of magnification conditions, exhibits the smallest possible errors of the image on the axis. For objectives of very long focal length, the weight of which is considerable, it is advisable to choose the form in which the flint is in front, as the power of resisting bending is then greater. Optical instruments to be used for the observation of either near or distant objects, as, for example, reading microscopes, should have an objective in which the spherical aberration in both directions has been removed. Finally, in cases where the light is monochromatic, the errors of chromatism and chromatic magnification are of less importance.

As already mentioned, the choice of the type of glass has an important influence upon the form of the objective. In order to give at least some general idea of the effect, tables and corresponding diagrams have been constructed for two special cases in which there is indicated the effect on the focal lengths and radii of the individual lenses arising from an alteration of the optical constants of the glasses used, and which enable the radii to be determined of objectives made from any types of glasses which lie within the limits of the tables.

One of these tables, C, p. 144, which deals with cemented objectives, has been calculated from approximate formulæ. The other, D, p. 146, dealing with objectives for which the spherical aberration has been eliminated in both directions, is based upon rigid trigonometrical computation, taking the thicknesses into consideration. Both tables were originally constructed for use in the optical workshops of C. A. Steinheil & Sons. The arrangement and use of these tables and of the corresponding diagrams II, III, IV, and V (at end of book) will briefly be described.

Table C, from which are obtained the radii of the objectives whose inner surfaces fit one another, has been calculated from the following well-known approximate formulæ for the elimination of chromatic and spherical aberrations.

For the correction of chromatism at the axis, the equation to be satisfied is[†]

$$(18) \qquad 0 = \frac{\Delta n}{n-1} \cdot \frac{1}{f_{0,2}} + \frac{\Delta n'}{n'-1} \cdot \frac{1}{f_{4,6}},$$

where

$$(19) \qquad f_{0,2} = 1 - \frac{\Delta n}{\Delta n'} \cdot \frac{n'-1}{n-1},$$

$$(20) \qquad f_{4,6} = -\left(\frac{\Delta n'}{\Delta n} \cdot \frac{n-1}{n'-1} - 1\right),$$

and

$$(21) \qquad \frac{1}{f_{0,6}} = \frac{1}{f_{0,2}} + \frac{1}{f_{4,6}}.$$

For the elimination of spherical aberration the equation is[‡]

$$(22) \quad 0 = \frac{n(4n-1)\,\textcircled{3}}{8(n-1)^2(n+2)} + \frac{n'(4n'-1)}{8(n'-1)^2(n'+2)} \times \frac{f_{0,2}}{f_{4,6}}\left[\textcircled{4}\left(\frac{f_{0,2}}{f_{4,6}}\right)^2 - \frac{4(n'-1)^2}{4n'-1} \cdot \frac{f_{0,2}}{f_{0,6}}\right],$$

where ③ and ④ have values which satisfy the equations 23 and 24[§]

$$(23) \quad \frac{1}{r_2} = \frac{4+n-2n^2}{2(n-1)(n+2)a'_2} + \frac{n(2n+1)}{2(n-1)(n+2)a_0} \mp \frac{n\sqrt{4n-1}}{2(n-1)(n+2)} \cdot \frac{\sqrt{\textcircled{3}-1}}{f_{0,2}}.$$

[†] Klügel's *Analytische Dioptrik*, Leipzig, 1778, p. 125, par. 333.

[‡] Klügel, p. 126, par. 337.

[§] Klügel's *Dioptrik*, p. 75, par. 197.

$$(24)\quad \frac{1}{r_4} = \frac{n'(2n'+1)}{2(n'-1)(n'+2)a'_6} + \frac{4+n'-2n'^2}{2(n'-1)(n'+2)a_4} \pm \frac{n'\sqrt{4n'-1}}{2(n'-1)(n'+2)} \cdot \frac{\sqrt{\textcircled{4}-1}}{f_{4,6}}.$$

The third equation to be satisfied in order that the inner surfaces shall fit one another is

$$(25)\qquad r_2 = -r_4.\dagger$$

From equations 22, 23, 24, and 25, the values of ③ and ④ may be found. When these are known, the radii of the objective may be obtained from the equations

$$(26)\quad \frac{1}{r_0} = \frac{4+n-2n^2}{2(n-1)(n+2)} \cdot \frac{1}{a_0} + \frac{n(2n+1)}{2(n-1)(n+2)} \cdot \frac{1}{a'_2} \pm \frac{n\sqrt{4n-1}}{2(n-1)(n+2)} \cdot \frac{\sqrt{\textcircled{3}-1}}{f_{0,2}}$$

$$(27)\quad \frac{1}{r_2} = \frac{1}{(n-1)f_{0,2}} - \frac{1}{r_0} \qquad (28)\quad \frac{1}{r_4} = -\frac{1}{r_2}\dagger$$

$$(29)\quad \frac{1}{r_6} = \frac{1}{(n'-1)f_{4,6}} - \frac{1}{r_4}.$$

From these formulæ the radii of seven objectives have been calculated, for which the refractive indexes of the glasses used throughout were

$$n = 1{\cdot}5154 \text{ and } n' = 1{\cdot}6140.$$

For the first objective, the value of $\frac{\Delta n'}{\Delta n}$ was 2·2134, and for the succeeding objectives 2·1728, 2·0744, 2·0250, 1·9083, 1·7668, and 1·6224, respectively. The first horizontal group of seven columns of Table C, p. 144, gives the particulars of these objectives.

A further series of seven objectives has been calculated for the refractive indexes.

$$n = 1{\cdot}5254 \text{ and } n' = 1{\cdot}6140,$$

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† The authors have retained Klügel's sign convention, according to which all convex surfaces have positive radii and all concave surfaces negative radii.—ED.

and for values of $\frac{\Delta n'}{\Delta n}$ the same as those of the previous series. The particulars of these objectives are tabulated in the next horizontal group of seven columns. Similarly the particulars in the succeeding groups have been calculated for corresponding variations of n and n'.

From the data of these tables there have been prepared the curves contained in Plates II, III, IV, and V (at end of book), in which the ordinates represent the values of the expression $\frac{\Delta n}{\Delta n'} \cdot \frac{n'-1}{n-1}$ and the abscissæ the corresponding values of the radii, or similarly, their reciprocals. Of these curves Plate II serves for the determination of r_0, III for $r_2 = -r_4$,† and IV and V for $\frac{1}{r_6}$.

By interpolation there can be obtained from Table C, p. 144, or from the curves on Plates II–V, the radii of any cemented objective, the refractive indexes of which lie within the limits

$$n = 1{\cdot}5054 \text{ and } 1{\cdot}5254,$$
$$\text{and} \quad n' = 1{\cdot}6040 \text{ and } 1{\cdot}6240,$$

and for which the value of $\frac{\Delta n'}{\Delta n}$ lies between 2·2134 and 1·6224. In all cases the true focal length $f_{0,6} = 1$. For any other value of $f_{0,6}$ the radii must be multiplied by the value in question.

Once more it should be noted that in the computation of the objectives in this Table C, the thicknesses of the lenses have been neglected, approximate formulæ having been used. The objectives in question can, therefore, only be employed when this limitation is permissible.

From Table D, p. 146, there may be obtained the radii of objectives for which the spherical aberration is eliminated in both directions. As already stated, these calculations have been made by rigidly trigonometrical methods, taking into account the lens thicknesses.

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† The authors have retained Klügel's sign convention, according to which all convex surfaces have positive radii and all concave surfaces negative radii.—ED.

C. CEMENTED

DATA FOR CURVES ON

	1.	2.	3.
n	1·515 4	1·515 4	1·515 4
n'	1·614 0	1·614 0	1·614 0
$\Delta n' : \Delta n$	2·213 4	2·172 8	2·074 4
$\frac{\Delta n}{\Delta n'}\,\frac{n'-1}{n-1}$	0·538 23	0·548 28	0·574 28
r_0	0·590 21	0·522 05	0·454 60
$-r_2 = -r_4$	0·398 74	0·420 20	0·424 11
$1 : r_6$	0·609 56	0·403 01	0·160 87
n	1·525 4	1·525 4	1·525 4
n'	1·614 0	1·614 0	1·614 0
$\Delta n' : \Delta n$	2·213 4	2·172 8	2·074 4
$\frac{\Delta n}{\Delta n'}\,\frac{n'-1}{n-1}$	0·528 03	0·537 84	0·563 31
r_0	impossible	impossible	0·513 51
$-r_2 = -r_4$	impossible	impossible	0·414 75
$1 : r_6$	impossible	impossible	0·311 00
n	1·505 4	1·505 4	1·505 4
n'	1·614 0	1·614 0	1·614 0
$\Delta n' : \Delta n$	2·213 4	2·172 8	2·074 4
$\frac{\Delta n}{\Delta n'}\,\frac{n'-1}{n-1}$	0·548 90	0·559 09	0·585 60
r_0	0·482 84	0·459 56	0·414 54
$-r_2 = -r_4$	0·431 97	0·432 60	0·423 32
$1 : r_6$	0·333 20	0·242 65	0·060 24
n	1·515 4	1·515 4	1·515 4
n'	1·624 0	1·624 0	1·624 0
$\Delta n' : \Delta n$	2·213 4	2·172 8	2·074 4
$\frac{\Delta n}{\Delta n'}\,\frac{n'-1}{n-1}$	0·547 02	0·557 17	0·583 60
r_0	0·500 35	0·474 54	0·426 06
$-r_2 = -r_4$	0·437 68	0·439 71	0·432 43
$1 : r_6$	0·349 52	0·257 88	0·066 47
n	1·515 4	1·515 4	1·515 4
n'	1·604 0	1·604 0	1·604 0
$\Delta n' : \Delta n$	2·213 4	2·172 8	2·074 4
$\frac{\Delta n}{\Delta n'}\,\frac{n'-1}{n-1}$	0·529 49	0·539 32	0·564 90
r_0	impossible	impossible	0·495 30
$-r_2 = -r_4$	impossible	impossible	0·409 77
$1 : r_6$	impossible	impossible	0·290 83

DOUBLETS

PLATES II, III, IV, AND V

4.	5.	6.	7.
1·515 4	1·515 4	1·515 4	1·515 4
1·614 0	1·614 0	1·614 0	1·614 0
2·025 0	1·908 3	1·766 8	1·622 4
0·588 28	0·624 28	0·674 28	0·734 28
0·430 20	0·380 22	0·323 84	0·273 01
0·418 74	0·394 62	0·348 55	0·296 37
0·061 03	−0·172 04	−0·502 50	−1·126 42
1·525 4	1·525 4	1·525 4	1·525 4
1·614 0	1·614 0	1·614 0	1·614 0
2·025 0	1·908 3	1·766 8	1·622 4
0·577 05	0·612 36	0·661 41	0·710 46
0·476 23	0·414 04	0·352 06	0·298 71
0·416 61	0·400 83	0·359 59	0·309 99
0·178 27	−0·078 00	−0·400 54	−0·770 43
1·505 4	1·505 4	1·505 4	1·505 4
1·614 0	1·614 0	1·614 0	1·614 0
2·025 0	1·908 3	1·766 8	1·622 4
0·599 87	0·636 59	0·687 58	0·738 57
0·394 78	0·351 21	0·298 44	0·249 20
0·414 63	0·385 03	0·335 32	0·281 25
−0·029 88	−0·254 97	−0·602 17	−1·043 95
1·515 4	1·515 4	1·515 4	1·515 4
1·624 0	1·624 0	1·624 0	1·624 0
2·025 0	1·908 3	1·766 8	1·622 4
0·597 88	0·634 41	0·685 22	0·736 04
0·405 35	0·360 08	0·306 23	0·256 16
0·424 20	0·395 24	0·345 03	0·290 11
−0·019 52	−0·250 84	−0·590 08	−1·021 71
1·515 4	1·515 4	1·515 4	1·515 4
1·604 0	1·604 0	1·604 0	1·604 0
2·025 0	1·908 3	1·766 8	1·622 4
0·578 67	0·614 09	0·663 27	0·712 46
0·461 97	0·403 86	0·343 70	0·291 36
0·409 75	0·391 91	0·350 56	0·301 59
0·166 62	−0·082 95	−0·408 58	−0·786 52

D. ACHROMATIC DOUBLETS

CORRECTED FOR SPHERICAL ABERRATION IN BOTH DIRECTIONS

	1.	2.	3.	4.
n	1·512 0	1·512 0	1·512 0	1·512 0
n'	1·628 0	1·628 0	1·628 0	1·628 0
$\Delta n' : \Delta n$	2·100 0	1·950 0	1·800 0	1·650 0
$\frac{\Delta n'}{\Delta n}\frac{n-1}{n'-1}$	1·712 1	1·589 8	1·467 5	1·345 2
r_0	0·632 80	0·564 53	0·484 91	0·390 83
$-r_2$	0·320 89	0·286 85	0·247 29	0·200 88
$-r_4$	0·328 37	0·296 19	0·257 73	0·210 42
$1 : r_6$	0·805 15	0·678 38	0·489 29	0·192 82
n	1·532 0	1·532 0	1·532 0	1·532 0
n'	1·628 0	1·628 0	1·628 0	1·628 0
$\Delta n' : \Delta n$	2·100 0	1·950 0	1·800 0	1·650 0
$\frac{\Delta n'}{\Delta n}\frac{n-1}{n'-1}$	1·779 0	1·651 9	1·524 8	1·397 8
r_0	0·683 12	0·615 74	0·537 12	0·444 26
$-r_2$	0·352 90	0·318 61	0·278 72	0·231 74
$-r_4$	0·355 62	0·325 29	0·287 80	0·240 10
$1 : r_6$	0·760 51	0·628 14	0·446 15	0·188 69
n	1·512 0	1·512 0	1·512 0	1·512 0
n'	1·598 0	1·598 0	1·598 0	1·598 0
$\Delta n' : \Delta n$	2·100 0	1·950 0	1·800 0	1·650 0
$\frac{\Delta n'}{\Delta n}\frac{n-1}{n'-1}$	1·798 0	1·669 5	1·541 1	1·412 7
r_0	0·675 12	0·610 18	0·534 36	0·444 77
$-r_2$	0·342 06	0·309 61	0·271 86	0·227 45
$-r_4$	0·342 99	0·314 49	0·278 57	0·234 96
$1 : r_6$	0·813 27	0·679 76	0·506 00	0·232 79
n	1·532 0	1·532 0	1·532 0	1·532 0
n'	1·598 0	1·598 0	1·598 0	1·598 0
$\Delta n' : \Delta n$	2·100 0	1·950 0	1·800 0	1·650 0
$\frac{\Delta n'}{\Delta n}\frac{n-1}{n'-1}$	1·868 2	1·734 8	1·601 3	1·467 9
r_0	0·725 05	0·660 75	0·585 86	0·497 49
$-r_2$	0·374 27	0·341 48	0·303 49	0·258 63
$-r_4$	0·370 42	0·342 06	0·306 56	0·263 63
$1 : r_6$	0·764 64	0·640 58	0·480 43	0·232 19

CHAPTER IV

Oblique Images

Calculation of an Image Point lying out of the Axis

In the case or six of the objectives discussed in the previous chapter, calculations have been made of the image of an object point lying 6° out of the axis. For the example A (*a*), 1*a*, not only will the complete calculation be given, but the nature and qualities of the oblique image itself will also be determined from the results.

This objective, for which all the data concerning an image point in the axis are given in Table A (*a*), Case 1*a*, p. 118, has a focal plane corresponding with the axial image point, that is, the point of intersection of the yellow rim ray with the axis at a distance of 670·46 mm. from the last vertex. Through this objective there have been reckoned three oblique rays in the axial plane, which intersect the first surface of the objective, one at the vertex of the surface, and the two others as upper and lower rim rays respectively. All three rays lie in one plane, containing the optical axis and the object lying under the axis at an obliquity of 6°. These rays will be referred to as the principal and the upper and lower rim rays. The data and calculations are as follows:—

$d_1 = 8{\cdot}0$
$d_3 = 6{\cdot}685$
$d_5 = 5{\cdot}0$

$r_0 = +\ 67{\cdot}282\,2$
$r_2 = +153{\cdot}021\,0$
$r_4 = -114{\cdot}880\,6$
$r_6 = -300{\cdot}334\,3$

	Principal Ray.	Upper Rim Ray.	Lower Rim Ray.
h_0		+30·0	−30·0
$\log h_0$		1·477 121	1·477 121*n*
$\log (1 : r_0)$		8·172 100	8·172 100
$\log \sin \eta_0$		9·649 221	9·649 221*n*
η_0		26° 28′ 47·5″	−26° 28′ 47·5″
$-\tau_0 = -\alpha_0$	6° 0′ 0″	6 0 0	6 0 0
ϕ_0	6 0 0	32 28 47·5	−20 28 47·5
$\log \sin \phi_0$	9·019 235	9·729 977	9·543 917*n*
$\log (1 : n)$	9·818 710	9·818 710	9·818 710
$\log \sin \phi_0'$	8·837 945	9·548 687	9·362 627*n*
$-\phi_0'$	−3° 56′ 53·9″	−20° 42′ 59·0″	+13° 19′ 30·5″
ϕ_0	6 0 0	32 28 47·5	−20 28 47·5
δ_0	2 3 6·1	11 45 48·5	− 7 9 17·0
α_0	−6 0 0	− 6 0 0	− 6 0 0
α_0'	−3 56 53·9	5 45 48·5	−13 9 17·0
$\log \sin \alpha_0'$	8·837 945*n*	9·001 829	9·357 137*n*
$\log (1 : \sin \alpha_0')$	1·162 055*n*	0·998 171	0·642 863*n*
$\log \sin \phi_0'$	8·837 945	9·548 687	9·362 627*n*
$\log r_0$	1·827 900	1·827 900	1·827 900
$\log (a_0' - r_0)$	1·827 900*n*	2·374 758	1·833 390
$a_0' - r_0$	− 67·282 17	237·005 3	68·138 09
$r_0 - r_2 - d_1$	− 93·738 8	− 93·738 8	−93·738 8
$a_0' - r_2 - d_1$	−161·020 97	143·266 5	−25·600 79
$\log (a_0' - r_2 - d_1)$	2·206 883*n*	2·156 145	1·408 252*n*
$\log \sin \alpha_0'$	8·837 945*n*	9·001 829	9·357 137*n*
$\log (1 : r_2)$	7·815 249	7·815 249	7·815 249
$\log \sin \phi_2$	8·860 077	8·973 223	8·580 638
$\log n$	0·181 290	0·181 290	0·181 290
$\log \sin \phi_2'$	9·041 367	9·154 513	8·761 928

ϕ_2	4° 9′ 18·3″	5° 23′ 41·8″	2° 10′ 55·4″
$-\phi_2'$	−6 18 53·8	−8 12 20·9	− 3 18 48·8
δ_2	−2 9 35·5	−2 48 39·1	− 1 7 53·4
α_0'	−3 56 53·9	5 45 48·5	−13 9 17·0
α_2'	−6 6 29·4	2 57 9·4	−14 17 10·4
$\log \sin \alpha_2'$	9·026 965*n*	8·711 891	9·392 285*n*
$\log (1 : \sin \alpha_2')$	0·973 035*n*	1·288 109	0·607 715*n*
$\log \sin \phi_2'$	9·041 367	9·154 513	8·761 928
$\log r_2$	2·184 751	2·184 751	2·184 751
$\log (a_2' - r_2)$	2·199 153*n*	2·627 373	1·554 394*n*
$a_2' - r_2$	−158·180 5	424·007 0	− 35·842 15
$r_2 - r_4 - d_3$	261·216 6	261·216 6	261·216 6
$a_2' - r_4 - d_3$	103·036 1	685·223 6	225·374 45
$\log (a_2' - r_4 - d_3)$	2·012 989	2·835 832	2·352 905
$\log \sin \alpha_2'$	9·026 965*n*	8·711 891	9·392 285*n*
$\log (1 : r_4)$	7·939 753*n*	7·939 753*n*	7·939 753*n*
$\log \sin \phi_4$	8·979 707	9·487 476*n*	9·684 943
$\log (1 : n')$	9·792 210	9·792 210	9·792 210
$\log \sin \phi_4'$	8·771 917	9·279 686*n*	9·477 153
ϕ_4	5° 28′ 34·9″	−17° 53′ 34·4″	28° 57′ 14·8″
$-\phi_4'$	−3 23 26·6	10 58 35·8	−17 27 32·1
δ_4	2 5 8·3	− 6 54 58·6	11 29 42·7
α_2'	−6 6 29·4	2 57 9·4	−14 17 10·4
α_4'	−4 1 21·1	− 3 57 49·2	− 2 47 27·7
$\log \sin \alpha_4$	8·846 020*n*	8·839 628*n*	8·687 470*n*
$\log (1 : \sin \alpha_4')$	1·153 980*n*	1·160 372*n*	1·312 530*n*
$\log \sin \phi_4'$	8·771 917	9·279 686*n*	9·477 153
$\log r_4$	2·060 247*n*	2·060 247*n*	2·060 247*n*
$\log (a_4' - r_4)$	1·986 144	2·500 305*n*	2·849 930
$a_4' - r_4$	96·859 89	−316·449 9	707·831 6
$r_4 - r_6 - d_5$	180·453 7	180·453 7	180·453 7
$a_4' - r_6 - d_5$	277·313 59	−135·996 2	888·285 3
$\log (a_4' - r_6 - d_5)$	2·442 971	2·133 527*n*	2·948 552
$\log \sin \alpha_4'$	8·846 020*n*	8·839 628*n*	8·687 470*n*
$\log (1 : r_6)$	7·522 395*n*	7·522 395*n*	7·522 395*n*
$\log \sin \phi_6$	8·811 386	8·495 550*n*	9·158 417
$\log n'$	0·207 790	0·207 790	0·207 790
$\log \sin \phi_6'$	9·019 176	8·703 340*n*	9·366 207

$d_1 = 8{\cdot}0$
$d_3 = 6{\cdot}685$
$d_5 = 5{\cdot}0$

$r_0 = +67{\cdot}282\,2$
$r_2 = +153{\cdot}021\,0$
$r_4 = -114{\cdot}880\,6$
$r_6 = -300{\cdot}334\,3$

	Principal Ray.	Upper Rim Ray.	Lower Rim Ray.
ϕ_6	3° 42′ 49·5″	−1° 47′ 37·2″	8° 16′ 49·5″
$-\phi_6'$	−5 59 57·1	2 53 42·0	−13 26 15·0
δ_6,	−2 17 7·6	1 6 4·8	− 5 9 25·8
a_4'	−4 1 21·1	−3 57 49·2	− 2 47 27·7
a_6'	−6 18 28·7	−2 51 44·4	− 7 56 53·2
$\log \sin a_6'$	9·040 889*n*	8·698 417*n*	9·140 748*n*
$\log (1 : \sin a_6')$	0·959 111*n*	1·301 583*n*	0·859 252*n*
$\log \sin \phi_6'$	9·019 176	8·703 340*n*	9·366 207
$\log r_6$	2·477 605*n*	2·477 605*n*	2·477 605*n*
$\log (a_6' - r_6)$	2·455 892	2·482 528*n*	2·703 064
$a_6' - r_6$	285·688 0	− 303·758 2	504·735 7
$r_6 - v_6'$	−970·80	− 970·80	−970·80
$a_6' - v_6'$	−685·112 0	−1 274·558 2	−466·064 3
$\log (a_6' - v_6')$	2·855 762*n*	3·105 360*n*	2·668 446*n*
$\log \tan a_6'$	9·043 527*n*	8·698 959*n*	9·144 940*n*
$\log h_6'$	1·879 289	1·804 319	1·813 386
h_6'	75·733 67	63·726 34	65·070 78

By reckoning in this way these three rays through the objective, there are obtained the heights of intersection of the emergent rays with the vertical plane passing through the focus of the image on the axis. The heights in question are 63·726 for the upper rim ray, 75·734 for the principal ray, and 65·071 mm. for the lower rim ray. It is now necessary to bring the focal plane nearer to the objective by an amount 15·0016 mm., that is, to a distance of 955·7984 mm. from the last centre plane, in order that the upper and lower rim rays may intersect the new focal plane in one point. The calculation is as follows:—

	Principal Ray.	Upper Rim Ray.	Lower Rim Ray.
$a_6' - r_6$	285·688 0	− 303·758 2	504·735 7
$r_6 - v_6''$	−955·798 4	− 955·798 4	−955·798 4
$a_6' - v_6''$	−670·110 4	−1 259·557 0	−451·062 7
log $(a_6' - v_6'')$	2·826 146*n*	3·100 218*n*	2·654 237*n*
log tan a_6'	9·043 527*n*	8·698 959*n*	9·144 940*n*
log h_6''	1·869 673	1·799 177	1·799 177
h_6''	74·075 22	62·976 28	62·976 28

At the plane distant 955·798 mm. from the last centre plane, the height of intersection of the upper and lower rim rays is accordingly 62·976 mm., and the principal ray intersects the plane at a height of 74·075 mm. from the axis. There is accordingly coma† in the image, the error being *uncorrected*, since the upper and lower rim rays intersect one another at a distance 11·099 mm. *below* the principal ray.

The data for the calculation of rays lying out of the axial plane are obtained as follows:—

log r_0	1·827 900	log x	0·849 520
log sin ϕ_0	9·019 235	log $(1 : q_0)$	8·511 136
log tan τ_0	9·021 620*n*	log cos ζ_0	9·360 656
log $(1 : \sin \tau_0)$	0·980 765*n*		
log x	0·849 520		
log x^2	1·699 040	ζ_0	76° 44′ 10·7″
x^2	50·008 06	π_0	180 0 0
h_0^2	900	$\pi_0 - \zeta_0$	103 15 49·3
q_0^2	950·008 06		
log q_0^2	2·977 727	log sin $(\pi_0 - \zeta_0)$	9·988 258
log q_0	1·488 864	log cos $(\pi_0 - \zeta_0)$	9·360 656*n*

With the help of these data there will now be traced an oblique ray lying out of the axial plane, having the same incidence height as the principal ray, but incident at the periphery of the objective, and oriented 90° from the upper and lower rim rays.

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† Throughout the original the word "verzerrung" is used to express "distortion" in the general sense. The term coma is here introduced as that aberration is distinctly implied.—ED.

In the calculations the corresponding values for all four consecutive refractions have been arranged adjacent to one another in horizontal rows, to facilitate quick comparison. The data and computation of the oblique ray out of the axial plane are as follows:—

$$\tau_0 = -6^\circ,$$
$$h_0 = 30{\cdot}0 \text{ mm.},$$
$$\pi_0 - \zeta_0 = 103^\circ\ 15'\ 49{\cdot}3'',$$
$$q_0 = 30{\cdot}822\ 2.$$

$\log(\mp\sin\tau)$	9·019 235*n*	9·246 352*n*	9·256 468	8·828 959
$\log\cos(\pi-\zeta)$	9·360 656*n*	7·605 593	9·968 495	9·644 638
$\log\cos\lambda$	8·379 891	6·851 945*n*	9·224 963	8·473 597
λ	88° 37′ 32·8″	90° 2′ 26·7″	80° 20′ 10·4″	88° 17′ 41·2″
$\log\sin\lambda$	9·999 875	0·000 000	9·993 793	9·999 808
$\log(1:r)$	8·172 100	7·815 249	7·939 753	7·522 395
$\log q$	1·488 864	1·068 718	1·675 639	1·507 618
$\log\sin\phi$	9·660 839	8·883 967	9·609 185	9·029 821
$\log(1:n)$	9·818 710	0·181 290	9·792 210	0·207 790
$\log\sin\phi'$	9·479 549	9·065 257	9·401 395	9·237 611
ϕ	27° 15′ 22·8″	4° 23′ 25·8″	23° 59′ 32·9″	6° 8′ 55·0″
$-\phi'$	−17 33 31·1	−6 40 25·1	−14 35 44·5	−9 57 8·0
$\phi-\phi'$	9 41 51·7	−2 16 59·3	9 23 48·4	−3 48 13·0
λ	88 37 32·8	90 2 26·7	80 20 10·4	88 17 41·2
λ'	98 19 24·5	87 45 27·4	89 43 58·8	84 29 28·2
$\log\sin\lambda'$	9·995 401	9·999 667	9·999 995	9·997 990
$\log\cos\lambda'$	9·160 653	8·592 477	7·668 387	8·982 268
$\phi+\phi'$	44° 48′ 53·9″	11° 3′ 50·9″	38° 35′ 17·4″	16° 6′ 3·0″
$\log\sin(\phi+\phi')$	9·848 078	9·283 092	9·794 988	9·442 995
$\log\sin(\phi-\phi')$	9·226 470	8·600 295*n*	9·212 908	8·821 754*n*
$\log(1:\sin\phi)$	0·339 161	1·116 033	0·390 815	0·970 179
$\log(1:\sin\phi')$	0·520 451	0·934 743	0·598 605	0·762 389
$\log 2\cot 2\omega$	9·934 160	9·934 163*n*	9·997 316	9·997 317*n*
$\log r$	1·827 900	2·184 751	2·060 247	2·477 605
$\log\sin\phi'$	9·479 549	9·065 257	9·401 395	9·237 611
$\log(1:\sin\lambda')$	0·004 599	0·000 333	0·000 005	0·002 010
$\log q'$	1·312 048	1·250 341	1·461 647	1·717 226

$\log(\sin\lambda : \sin\lambda')$	0·004 474	0·000 333	9·993 798	0·001 818
$\log(1 : n)$	9·818 710	0·181 290	9·792 210	0·207 790
$\log q$	1·488 864	1·068 718	1·675 639	1·507 618
$\log q'$	1·312 048	1·251 341‡	1·461 647	1·717 226
$\log(\sin\lambda' : \sin\lambda)$	9·995 526	9·999 667	0·006 202	9·998 182
$\log\sin\tau$	9·019 235	9·246 352	9·256 468	8·828 959
$\log\sin(\pi-\zeta)$	9·988 258	9·999 997	9·565 250	9·952 990
$\log\{\sin\tau'\sin(\pi'-\zeta)\}$	9·003 019	9·246 016	8·827 920	8·780 131
$-\log(\mp\cos\lambda')$	−9·160 653	−8·592 477*n*	−7·668 387	−8·982 268
$\log\tan(\pi'-\zeta)$	9·842 366	0·653 539*n*	1·159 533	9·797 863
$(\pi'-\zeta)$	34° 49′ 22·3″	102° 31′ 10·9″	86° 2′ 17·4″	32° 7′ 23·2″
ζ	76 44 10·7	21 47 24·8	102 44 55·6	124 58 3·2
π'	111 33 33·0	124 18 35·7	188 47 13·0	157 5 26·4
π	180	111 33 33·0	124 18 35·7	188 47 13·0
$\pi-\pi'$	68 26 27·0	−12 45 2·7	−64 28 37·3	31 41 46·6
$\log\sin(\pi'-\zeta)$	9·756 667	9·989 548	9·998 961	9·725 700
$\log\cos(\pi'-\zeta)$	9·914 302	9·336 010*n*	8·839 427	9·927 836
$\log\{\sin\tau'\sin(\pi'-\zeta)\}$	9·003 019	9·246 016	8·827 920	8·780 131
$\log\{1 : \sin(\pi'-\zeta)\}$	0·243 333	0·010 452	0·001 039	0·274 300
$\log\sin\tau'$	9·246 352	9·256 468	8·828 959	9·054 431
$\log\tan\tau$	9·021 620	9·253 211	9·263 660	8·829 949
$\log\sin(\pi-\zeta)$	9·988 258	9·999 997	9·565 250	9·952 990
$\log\{1 : \sin(\pi'-\zeta)\}$	0·243 333	0·010 452	0·001 039	0·274 300
$\log\tan\tau'$	9·253 211	9·263 660	8·829 949	9·057 239
$\log\sin\lambda$	9·999 875	0·000 000	9·993 793	9·999 808
$\log\sin\tau'$	9·246 352	9·256 468	8·828 959	9·054 431
$\log\{1 : \sin(\pi-\zeta)\}$	0·011 742	0·000 003	0·434 750	0·047 010
	9·257 969	9·256 471	9·257 502	9·101 249
$\log\sin\lambda'$	9·995 401	9·999 667	9·999 995	9·997 990
$\log\sin\tau$	9·019 235	9·246 352	9·256 468	8·828 959
$\log\{1 : \sin(\pi'-\zeta)\}$	0·243 333	0·010 452	0·001 039	0·274 300
	9·257 969	9·256 471	9·257 502	9·101 249
$\log\sin(\phi-\phi')$	9·226 470	8·600 295*n*	9·212 908	8·821 754*n*
$-\log\sin(\pi-\pi')$	−9·968 501	−9·343 822	−9·955 405*n*	−9·720 504
	9·257 969	9·256 473*n*	9·257 503	9·101 250*n*

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‡ Owing to an error of addition this number is wrong; it should read 1·250341. As the error is repeated and affects the whole of the subsequent calculations and discussions, it has been considered advisable to leave it uncorrected in preference to making the extensive alteration of the original figures that would be involved. When the correction is made $\xi = -0° 21' 57{\cdot}6''$ instead of $-0° 18' 11{\cdot}4''$, and $s = 72{\cdot}013$ mm. instead of $72{\cdot}1355$ mm.—Ed.

$$\tau_0 = -6^\circ,$$
$$h_0 = 30{\cdot}0 \text{ mm.},$$
$$\pi_0 - \zeta_0 = 103^\circ\ 15'\ 49{\cdot}3'',$$
$$q_0 = 30{\cdot}822\ 2.$$

c	93·738 8	−261·216 6	−180·453 7	970·80
$\log q'$	1·312 048	1·251 341	1·461 647	1·717 226
$\log\cos(\pi'-\zeta)$	9·914 302	9·336 010	8·839 427	9·927 836
$\log q'\cos(\pi'-\zeta)$	1·226 350	0·587 351 *n*	0·301 074	1·645 062
$\log(-c)$	1·971 919*n*	2·417 001	2·256 366	2·987 130*n*
$\log\tan\tau'$	9·253 211	9·263 660	8·829 949	9·057 239
$\log(-c\tan\tau')$	1·225 130*n*	1·680 661	1·086 315	2·044 369*n*
$q'\cos(\pi'-\zeta)$	16·840 31	−3·866 794	2·000 203	44·163 35
$-c\tan\tau'$	−16·793 07	47·935 91	12·198 74	−110·756 5
$q_2\cos(\pi'-\zeta_2)$	0·047 24	44·069 11	14·198 94	− 66·593 15
$\log q'$	1·312 048	1·251 341	1·461 647	1·717 226
$\log\sin(\pi'-\zeta)$	9·756 667	9·989 548	1·998 961	9·725 700
$\log\{q_2\sin(\pi'-\zeta_2)\}$	1·068 715	1·240 889	1·460 608	1·442 926
$-\log\{q_2\cos(\pi'-\zeta_2)\}$	−8·674 310	−1·644 134	−1·152 256	−1·823 430*n*
$\log\tan(\pi'-\zeta_2)$	2·394 405	9·596 755	0·308 352	9·619 496*n*
$\pi'-\zeta_2$	89° 46′ 8·2″	21° 33′ 40·1″	63° 49′ 9·8″	157° 23′ 37·8″
π'	111 33 33·0	124 18 35·7	188 47 13·0	157 5 26·4
ζ_2	21 47 24·8	102 44 55·6	124 58 3·2	− 0 18 11·4
ζ	76 44 10·7	21 47 24·8	102 44 55·6	124 58 3·2
$\zeta-\zeta_2$	54 56 45·9	−80 57 30·8	−22 13 7·6	125 16 14·6
$\log\sin(\pi'-\zeta_2)$	9·999 997	9·565 250	9·952 990	9·584 777
$\log\cos(\pi'-\zeta_2)$	7·605 593	9·968 495	9·644 638	9·965 281 *n*
$\log\{q_2\sin(\pi'-\zeta_2)\}$	1·068 715	1·240 889	1·460 608	1·442 926
$\log\{1:\sin(\pi'-\zeta_2)\}$	0·000 003	0·434 750	0·047 010	0·415 223
$\log q_2$	1·068 718	1·675 639	1·507 618	1·858 149
$\log\{1:\sin(\pi'-\zeta)\}$	0·243 333	0·010 452	0·001 039	0·274 300
	1·312 051	1·686 091	1·508 657	2·132 449
$\log(c\tan\tau')$	1·225 130	1·680 661 *n*	1·086 315*n*	2·044 369
$-\log\sin(\zeta-\zeta_2)$	−9·913 079	−9·994 570*n*	−9·577 657 *n*	9·911 920
	1·312 051	1·686 091	1·508 658	2·132 449
$\log q'$	1·312 048	1·251 341	1·461 647	1·717 226
$\log\{1:\sin(\pi'-\zeta_2)\}$	0·000 003	0·434 750	0·047 010	0·415 223
	1·312 051	1·686 091	1·508 657	2·132 449

At the focal plane situated 970·80 mm. from the last centre plane, the computation gives for the oblique image the value

$$\zeta_8 = \xi = -0^\circ\ 18'\ 11{\cdot}4'' \quad \text{and} \quad q_8 = z = 72{\cdot}1355 \text{ mm.}$$

From these data it will be found, by means of the following calculation, that the height above the axis of the point of intersection of the ray with the chosen focal plane is 72·135 mm.:—

$$\begin{aligned} \log z &= 1{\cdot}858\,149 \\ \log \cos \xi &= \underline{9{\cdot}999\,999} \\ \log (z \cos \xi) &= 1{\cdot}858\,148 \\ x = z \cos \xi &= 72{\cdot}135\,33 \end{aligned}$$

and from the following calculation, that the separation of the two side rays symmetrically placed relatively to the vertical plane containing the axis is 0·7634 mm., the distance of each ray from the axial plane being 0·3817 mm.:—

$$\begin{aligned} \log 2 &= 0{\cdot}301\,030 \\ \log z &= 1{\cdot}858\,149 \\ \log \sin \xi &= \underline{7{\cdot}723\,557n} \\ \log (2\, z \sin \xi) &= 9{\cdot}882\,736n \end{aligned}$$

This distance is reckoned as positive when the rays intersect behind the chosen focal plane, and as negative when the intersection lies before it. In the first case the residual error is uncorrected, in the second case over-corrected.

The intersection of the two symmetrically situated side rays takes place in an image plane which is nearer the objective by an amount 8·592 mm., and which is accordingly situated at a distance of 962·208 mm. from the last centre plane. This image plane may be determined from the calculation:

$$\begin{aligned} \log q_6' &= 1{\cdot}717\,226 \\ \log \sin \zeta_6 &= 9{\cdot}913\,537 \\ \log (1 : \tan \tau_6') &= 0{\cdot}942\,761 \\ \log (1 : \sin \pi_6') &= \underline{0{\cdot}409\,745} \\ & \ 2{\cdot}983\,269 \end{aligned}$$

The height of the point of intersection of the principal ray at this focal plane is determined thus:

$$\begin{array}{rcr}
a_6' - r_6 & = & 285{\cdot}688\,0 \\
r_6 - v_6'' & = & -962{\cdot}208\,0 \\
\hline
a_6' - v_6'' & = & -676{\cdot}520\,0 \\
\log\,(a_6' - v_6'') & = & 2{\cdot}830\,281\,n \\
\log\tan\alpha_6' & = & 9{\cdot}043\,527\,n \\
\hline
\log h_6'' & = & 1{\cdot}873\,808 \\
h_6'' & = & 74{\cdot}783\,88
\end{array}$$

and the height of the point of intersection of the two side rays is reckoned as follows:—

$$\begin{array}{rcr}
c_7 & = & 962{\cdot}208\,0 \\
\log\,(-c_7) & = & 2{\cdot}983\,269\,n \\
\log\tan\tau_6' & = & 9{\cdot}057\,239 \\
\hline
\log\,(-c_7\tan\tau_6') & = & 2{\cdot}040\,508\,n
\end{array}$$

$$\begin{array}{rcr}
q_6'\cos\,(\pi_6' - \zeta_6) & = & 44{\cdot}163\,35 \\
-c_7\tan\tau_6' & = & -109{\cdot}776\,15 \\
\hline
s\cos\,(\pi_6' - \xi) & = & -\;65{\cdot}612\,80
\end{array}$$

$$\begin{array}{rcr}
\log\,\{s\sin\,(\pi_6' - \xi)\} & = & 1{\cdot}442\,926 \\
-\log\,\{s\cos\,(\pi_6' - \xi)\} & = & -1{\cdot}816\,989\,n \\
\hline
\log\tan\,(\pi_6' - \xi) & = & 9{\cdot}625\,937\,n
\end{array}$$

$$\begin{array}{rcrrr}
\pi_6' - \xi & = & 157^\circ & 5' & 26{\cdot}6'' \\
\pi_6' & = & 157 & 5 & 26{\cdot}4 \\
\hline
-\xi & = & 0 & 0 & 0{\cdot}2 \\
\zeta_6 & = & 124 & 58 & 3{\cdot}2 \\
\hline
\zeta_6 - \xi & = & 124 & 58 & 3{\cdot}4
\end{array}$$

$$\begin{array}{rcr}
\log\sin\,(\pi_6' - \xi) & = & 9{\cdot}590\,254 \\
\log\cos\,(\pi_6' - \xi) & = & 9{\cdot}964\,317\,n
\end{array}$$

$$\begin{array}{rcr}
\log\,\{s\sin\,(\pi_6' - \xi)\} & = & 1{\cdot}442\,926 \\
\log\,\{1 : \sin\,(\pi_6' - \xi)\} & = & 0{\cdot}409\,746 \\
\hline
\log s & = & 1{\cdot}852\,672 \\
\log\,\{1 : \sin\,(\pi_6' - \zeta_6)\} & = & 0{\cdot}274\,300 \\
\hline
 & & 2{\cdot}126\,972
\end{array}$$

$$\begin{array}{rcr}
\log\,(c_7\tan\tau_6') & = & 2{\cdot}040\,508 \\
-\log\sin\,(\zeta_6 - \xi) & = & -9{\cdot}913\,536 \\
\hline
 & & 2{\cdot}126\,972
\end{array}$$

$$
\begin{aligned}
\log q_6' &= 1{\cdot}717\,226 \\
\log\{1 : \sin(\pi_6' - \xi)\} &= \underline{0{\cdot}409\,746} \\
&\ 2{\cdot}126\,972
\end{aligned}
$$

$$
\begin{aligned}
z &= 71{\cdot}231\,5 \\
\log\cos\xi &= 0{\cdot}000\,000 \\
\log z &= \underline{1{\cdot}852\,672} \\
\log x &= 1{\cdot}852\,672 \\
x &= 71{\cdot}231\,5
\end{aligned}
$$

The upper and lower rim rays met in a plane whose distance from the last centre plane is 955·7984 mm., while the two side rays intersect in the plane whose distance from the last centre plane is 962·2080 mm. As the focal plane for the image point lying out of the axis there will be chosen that plane whose distance from the last centre plane is the arithmetical mean of the two values, namely, 959·0032 mm. For this new focal plane there should be determined the intersection points of the five rays in question, and firstly for the three oblique rays in the axial plane; that is, the principal ray, and the upper and lower rim rays:

	Principal Ray.	Upper Rim Ray.	Lower Rim Ray.
$a_6' - r_6$	285·688 0	− 303·758 2	504·735 7
$r_6 - v_6'''$	−959·003 2	− 959·003 2	−959·003 2
$a_6' - v_6'''$	−673·315 2	−1 262·761 4	−454·267 5
$\log(a_6' - v_6''')$	2·828 218*n*	3·101 321*n*	2·657 312*n*
$\log(\tan\alpha_6')$	9·043 527*n*	8·698 959*n*	9·144 940*n*
$\log h_6'''$	1·871 745	1·800 280	1·802 252
h_6'''	74·429 48	63·136 43	63·423 76

and then as follows for the two oblique side rays:—

$$
\begin{aligned}
c_7 &= 959{\cdot}003\,2 \\
\log(-c_7) &= 2{\cdot}981\,820n \\
\log\tan\tau_6' &= \underline{9{\cdot}057\,239} \\
\log(-c_7\tan\tau_6') &= 2{\cdot}039\,059n
\end{aligned}
$$

$$\begin{array}{rcl}
q_6' \cos(\pi_6' - \zeta_6) & = & 44{\cdot}163\,35 \\
-c_7 \tan \tau_6' & = & -109{\cdot}410\,5 \\ \cline{3-3}
z \cos(\pi_6' - \xi) & = & -\ 65{\cdot}247\,15
\end{array}$$

$$\begin{array}{rcl}
\log z \sin(\pi_6' - \xi) & = & 1{\cdot}442\,926 \\
-\log z \cos(\pi_6' - \xi) & = & -1{\cdot}814\,562n \\ \cline{3-3}
\log \tan(\pi_6' - \xi) & = & 9{\cdot}628\,364n
\end{array}$$

$$\begin{array}{rcl}
\pi_6' - \xi & = & 156^\circ\ 58'\ 32{\cdot}5'' \\
\pi_6' & = & 157\quad 5\ \ 26{\cdot}4 \\ \cline{3-3}
\xi & = & 0\quad 6\ \ 53{\cdot}9 \\
\zeta_6 & = & 124\ \ 58\quad 3{\cdot}2 \\ \cline{3-3}
\zeta_6 - \xi & = & 124\ \ 51\quad 9{\cdot}3
\end{array}$$

$$\begin{array}{rcl}
\log \sin(\pi_6' - \xi) & = & 9{\cdot}592\,312 \\
\log \cos(\pi_6' - \xi) & = & 9{\cdot}963\,948n
\end{array}$$

$$\begin{array}{rcl}
\log \{z \sin(\pi_6' - \xi)\} & = & 1{\cdot}442\,926 \\
\log \{1 : \sin(\pi_6' - \xi)\} & = & 0{\cdot}407\,688 \\ \cline{3-3}
\log z & = & 1{\cdot}850\,614 \\
\log \{1 : \sin(\pi_6' - \zeta_6)\} & = & 0{\cdot}274\,300 \\ \cline{3-3}
 & & 2{\cdot}124\,914
\end{array}$$

$$\begin{array}{rcl}
\log (c_7 \tan \tau_6') & = & 2{\cdot}039\,059 \\
-\log \sin(\zeta_6 - \xi) & = & -9{\cdot}914\,145 \\ \cline{3-3}
 & & 2{\cdot}124\,914
\end{array}$$

$$\begin{array}{rcl}
\log q_6' & = & 1{\cdot}717\,226 \\
\log \{1 : \sin(\pi_6' - \xi)\} & = & 0{\cdot}407\,688 \\ \cline{3-3}
 & & 2{\cdot}124\,914
\end{array}$$

$$\begin{array}{rcl}
\log 2 & = & 0{\cdot}301\,030 \\
\log z & = & 1{\cdot}850\,614 \\
\log \sin \xi & = & 7{\cdot}302\,570 \\ \cline{3-3}
\log 2y & = & 9{\cdot}454\,214
\end{array}$$

$$\begin{array}{rcl}
2y & = & 0{\cdot}284\,586 \text{ over-corrected.} \\
y & = & \pm 0{\cdot}142 \\
\log z & = & 1{\cdot}850\,614 \\
\log \cos \xi & = & 9{\cdot}999\,999 \\ \cline{3-3}
\log x & = & 1{\cdot}850\,613 \\
x & = & 70{\cdot}895
\end{array}$$

The results thus determined for this focal plane are accordingly as follows: the upper rim ray intersects at a height of 63·136 mm., the principal ray at a height of 74·429 mm., the lower rim ray at a height of 63·434 mm. above the axis; the two side rim rays intersect at a height of 70·895 mm., and at distances of ± 0·142 mm. towards the right and left of the vertical axial plane. The sign ± is used to indicate that the rays out of the axial plane meet the chosen focal plane *before* they intersect one another, whereas the sign ∓ indicates that they meet the plane *after* their intersection. By means of the results so far obtained, the form of the image of an object point lying out of the axis may be represented diagrammatically, indicating on the one hand the principal ray, and on the other hand the two rim rays in the axial plane and those out of the axial plane. Thus, in the table on p. 165, the principal ray is indicated by the number 1. Of the rim rays oriented from each other by 90°, the upper is denoted by 2, the right-hand ray by 3, the lower by 4, and the left-hand ray by 5. In addition to these five rays, all of the same colour, there has been calculated a principal ray of greater refraction which is incident in the same line as the yellow principal ray. In the above example this second principal ray intersects the image plane at a greater height (1*). Finally, there has been determined the radius of the sphere which can be drawn through the image point on the axis and the image point lying out of the axis. The radius of this sphere is a measure of the image curvature. For the objective in question, the radius of curvature is determined as follows:—

$$(30)\quad (r) = \frac{(74{\cdot}429)^2 + (970{\cdot}7982 - 959{\cdot}0032)^2}{2 \times 11{\cdot}7950} = 240\ 732 \text{ mm.}$$

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Table of Aberrations of the Oblique Image Points for a Series of Objectives

Five additional objectives have been computed in the same manner as in the case of the above objective, and the six examples have been tabulated on pp. 163–164. The values in the individual horizontal rows are as follows:—

The headings give the radii and thicknesses of the objectives.

Row I indicates the focal plane of the axial image, the position of which is measured outwards from the vertex of the last lens surface.

Row II refers to the oblique rays in the axial plane, and indicates the focal plane in which the upper and lower rays meet, and also the heights at which the principal ray and the two rim rays intersect this plane. There is also indicated the distance between these points of intersection, that is, the coma.

Row III refers to the oblique rays not in the axial plane, and indicates the focal plane in which the side rays meet, and the heights at which the principal ray and the side rays meet this plane.

Row IV refers to the mean plane lying midway between the two preceding focal planes, its distance being reckoned from the vertex of the last surface of the objective. It also gives the points of intersection with this plane of the principal ray, and of the upper and lower and the two side rays.

In the last two rows there are indicated the astigmatism, that is, the distance of the focal plane in Row II from the focal plane in Row III, and the radius of curvature of the image respectively.

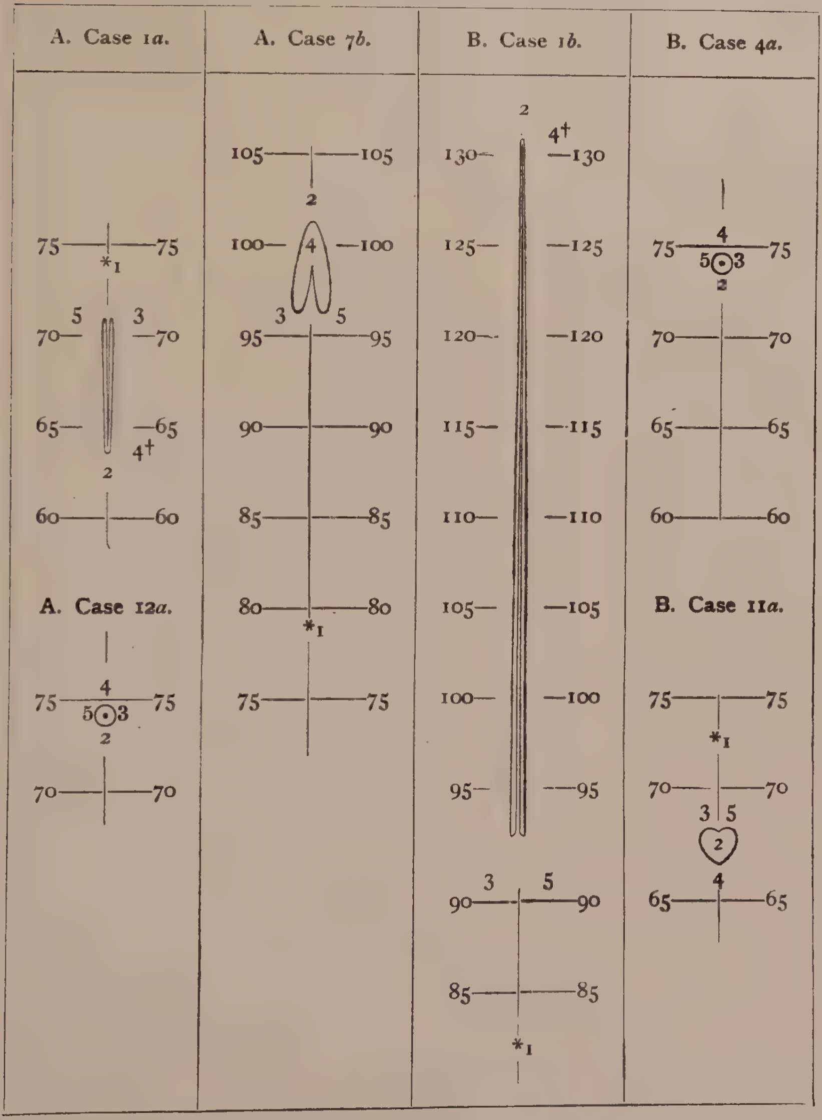
		A: Case 1*a*.	A. Case 12*a*.	A. Case 7*b*.
		$d_1 = 8{\cdot}0$ $r_0 = +\ 67{\cdot}282$ $d_3 = 6{\cdot}68$ $r_2 = +153{\cdot}02$ $d_5 = 5{\cdot}0$ $r_4 = -114{\cdot}88$ $r_6 = -300{\cdot}33$	$d_1 = 8{\cdot}0$ $r_0 = +441{\cdot}45$ $d_3 = 0{\cdot}01$ $r_2 = -196{\cdot}12$ $d_5 = 5{\cdot}0$ $r_4 = -201{\cdot}12$ $r_0 = -977{\cdot}97$	$d_1 = 8{\cdot}0$ $r_0 = -976{\cdot}63$ $d_3 = 0{\cdot}01$ $r_2 = -100{\cdot}95$ $d_5 = 5{\cdot}0$ $r_4 = +\ 69{\cdot}048$ $r_6 = +\ 48{\cdot}573$
I. Axial image ...	Focal plane	670·46	713·81	691·80
II. Oblique rays in the axial plane	Focal plane	655·46	699·28	748·67
	Heights: Yellow principal ray	74·075	74·150	81·875
	Heights: Intersecting rim rays	62·976	74·152	103·57
	Heights: Violet principal ray	74·193	74·150	81·955
	Coma	11·099 u.c.*	0·001 7 o.c.†	21·690 o.c.†
III. Oblique rays not in the axial plane	Focal plane	661·87	706·95	697·31
	Heights: Principal ray ...	74·784	74·955	76·318
	Heights: Intersecting side rays	71·231	74·957	81·830
IV. Mean plane between Planes II and III... ...	Focal plane	658·67	703·10	722·99
	Heights in the axial plane: Upper rim ray	63·136	74·389	100·95
	Heights in the axial plane: Principal ray	74·429	74·550	79·096
	Heights in the axial plane: Lower rim ray	63·434	74·715	99·114
	Out of the axial plane: Height	70·895	74·554	95·850
	Out of the axial plane: Horizontal displacement ...	± 0·142 3	± 0·161 2	∓ 0·990 8
V. Astigmatism ...	——	6·410 o.c.†	7·70 o.c.†	51·359 u.c.*
VI. Radius of curvature of the image	——	+240·73	+264·82	−115·88

* u.c. means uncorrected.   † o.c. means over-corrected.

		B. Case 1*b*.	B. Case 4*a*.	B. Case 11*a*.
		$d_1 = 5{\cdot}0$ $d_3 = 13{\cdot}93$ $d_5 = 8{\cdot}0$ $r_0 = +88{\cdot}961$ $r_2 = +54{\cdot}143$ $r_4 = -77{\cdot}827$ $r_6 = -45{\cdot}457$	$d_1 = 5{\cdot}0$ $d_3 = 0{\cdot}01$ $d_5 = 8{\cdot}0$ $r_0 = +\ 308{\cdot}20$ $r_2 = +\ 138{\cdot}18$ $r_4 = +\ 136{\cdot}13$ $r_6 = -17\,654{\cdot}1$	$d_1 = 5{\cdot}0$ $d_3 = 0{\cdot}01$ $d_5 = 8{\cdot}0$ $r_0 = -105{\cdot}73$ $r_2 = -193{\cdot}78$ $r_4 = +187{\cdot}84$ $r_6 = -453{\cdot}67$
I. Axial image ...	Focal plane	788·77	710·41	732·06
II. Oblique rays in the axial plane	Focal plane	848·58	695·93	718·21
	Heights: Yellow principal ray	81·144	74·155	74·218
	Heights: Intersecting rim rays	132·39	74·150	69·925
	Heights: Violet principal ray	80·874	74·157	74·165
	Coma	51·25 o.c.*	0·005 o.c.*	4·293 u.c.†
III. Oblique rays not in the axial plane	Focal plane	831·76	703·74	677·18
	Heights: Principal ray ...	79·584	74·972	70·028
	Heights: Intersecting side rays	89·924	74·966	67·718
IV. Mean plane between Planes II and III... ...	Focal plane	840·17	699·84	697·69
	Heights in the axial plane: Upper rim ray	131·391	74·400	68·856
	Heights in the axial plane: Principal ray	80·364	74·561	72·123
	Heights in the axial plane: Lower rim ray	130·812	74·733	67·076
	Out of the axial plane: Height	90·811	74·557	69·042
	Out of the axial plane: Horizontal displacement ...	∓ 0·295	± 0·163	∓ 0·953
V. Astigmatism ...	——	16·82 u.c.†	7·81 o.c.*	41·029 u.c.†
VI. Radius of curvature of the image	——	−88·524	+268·26	+92·856

* o.c. means over-corrected. † u.c. means uncorrected.

DIFFUSED IMAGES (MAGNIFIED 2½ FOLD)



1 = Principal ray.
2 = Upper rim ray.
3 = Right-side rim ray.
4 = Lower rim ray.
5 = Left-side rim ray.

† In Cases A. 1*a* and B. 1*b*, the number 4 is placed at the side of the position to which it refers.

Discussion of the Aberrations of the Image of an Object Point lying out of the Axis, and the Relationship of these Errors to the Principal Points

Of the objectives selected for the computation of image points lying out of the axis, three have the crown lens in front and three the flint. The examples of each set are so arranged that in the first the radius r_0 is strongly positive, in the second r_0 is moderately positive, and in the third the first surface is hollow. The second objective of each set exhibits always small defects of the image, and it should also be noted that there are considerable differences between the focal lengths of the axial and rim rays, and that the differences are of opposite

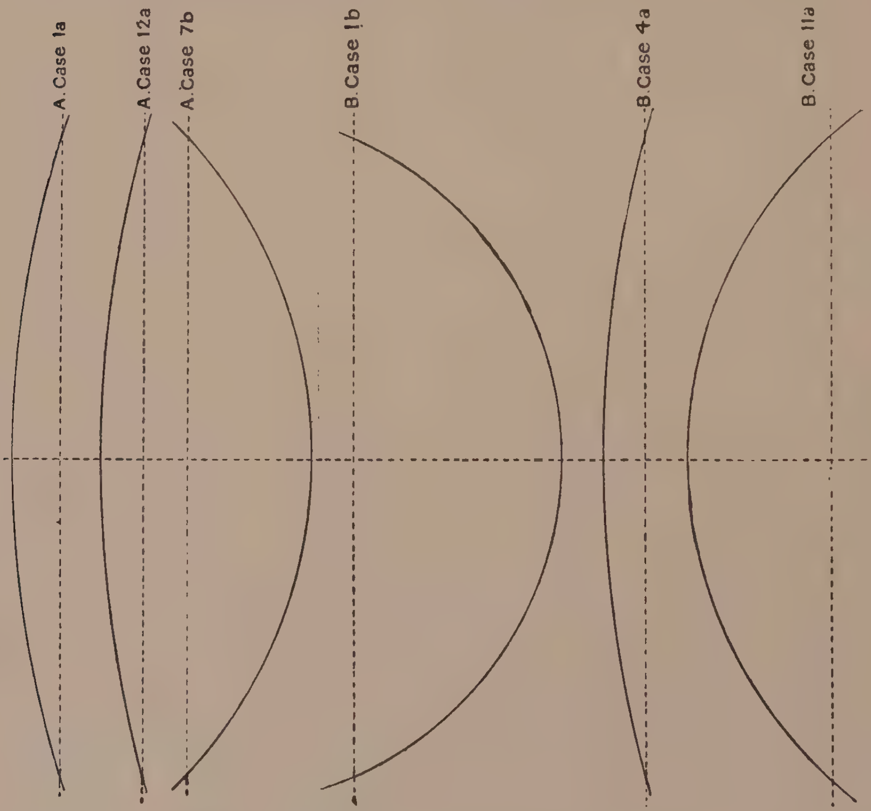


Fig. 20.—Curvature of the Image Surfaces. Magnified 5-fold

sense in the two sets. Those examples in which the principal points are widely separated for different colours have not been computed, since, in order to make these errors large, it is necessary to choose considerable separations of the lenses. The effect of the lens separations and also the separations of the principal points for rays of different colours in the case of complicated systems of lenses cannot be dealt with fully in this volume.

If now the aberrations *in the axis* are compared with the results of the computations for those *out of the axis*, it will be seen that when the focal length of the rim ray is smaller than that of the paraxial ray, the oblique rim rays both in the axial plane and out of it intersect their respective focal planes at a height *smaller* than does the principal ray. When the focal length of the rim ray is greater than that of the paraxial ray, the oblique rim rays both in the axial plane and out of it intersect their focal plane at a *greater* height than does the principal ray. The truth of these statements will be evident from the following table:—

$({}_Df_{0,6})_{\text{rim}}$ less than $({}_Df_{0,6})_{\text{axial}}$

Example.	A. Case 1*a*.		B. Case 11*a*.		B. Case 4*a*.	
Ray (yellow).	Axial.	Rim.	Axial.	Rim.	Axial.	Rim.
Focal length ...	720·00	685·20	720·00	706·47	720·00	720·00
Ray (yellow).	Principal.	Oblique Rim.	Principal.	Oblique Rim.	Principal.	Oblique Rim.
Intersection height:						
In the axial plane	74·075	62·976	74·218	69·925	74·155	74·150
Out of the plane	74·784	71·231	70·028	67·718	74·972	74·966

$({}_Df_{0,6})_{\text{rim}}$ greater than $({}_Df_{0,6})_{\text{axial}}$

Example.	A. Case 7*b*.		B. Case 1*b*.		A. Case 12*a*.	
Ray (yellow).	Axial.	Rim.	Axial.	Rim.	Axial.	Rim.
Focal length ...	720·01	770·20	720·00	811·98	720·00	720·01
Ray (yellow).	Principal.	Oblique Rim.	Principal.	Oblique Rim.	Principal.	Oblique Rim.
Intersection height:						
In the axial plane	81·875	103·57	81·144	132·39	74·150	74·152
Out of the plane	76·318	81·83	79·584	89·924	74·955	74·957

When the focal lengths of the rim and axial rays are different, the sine condition is not completely fulfilled, and there is coma present; whereas when the focal lengths of the rim and axial rays are equal, or nearly so, the rays both in and out of the axial plane intersect on the principal ray; that is, the image is free from coma.

The values of the coma are as follows:—

Example.	A. Case 1*a*.	B. Case 11*a*.	B. Case 4*a*.
Coma... ...	11·099 uncorrected.	4·293 uncorrected.	0·005 uncorrected.
Example.	A. Case 7*b*.	B. Case 1*b*.	A. Case 12*a*.
Coma... ...	21·70 over-corrected.	51·25 over-corrected.	0·002 over-corrected.

In those objectives that satisfy the sine condition, the image surface is concave to the objective. By altering the form of the lenses the curvature of the image can be altered in both directions, according as the effect of the positive or the negative lens preponderates.

From the above discussion it also follows that definition of the image point in the axis is obtained by coincidence of the *focal* points and definition of the image point out of the axis by coincidence of the *principal* points, and that although the definition at the axis may be very good, the aberrations of the image out of the axis may be very great, which can be most readily seen from a consideration of the diagrams in the table on p. 163, showing the distribution of the rays in the image point out of the axis.

APPENDIX I

Concerning the Determination of the Refraction and Dispersive Ratios of Various Media

BY

C. A. von STEINHEIL and L. von SEIDEL

[Reproduced from the *Transactions, Class II, of the Imperial Bavarian Academy of Science*, Volume V, Section II, p. 1. The notation alone has been altered to correspond with that introduced by us.]

The beautiful discovery of the fixed lines in the solar spectrum, which alone has made possible an exact determination of the refraction ratios for any definite ray of light, has found less general application than might have been expected to result from this excellent work of Fraunhofer. The reason for this may be due partly to the costly nature of the apparatus, which few possess, used by him in making the measurements, and partly to the fact that Fraunhofer himself drew attention to a difference between the medium dispersions deduced from his observations and those found by him to give the best effect, thus throwing doubt upon the theoretical results.

Although the Fraunhofer apparatus completely fulfilled its purpose for the determination of the refraction ratios, the observer cannot fail to realize that the finite distance of the source of light involved awkward calculations, and that a very considerable amount of space was required. We shall now endeavour to show that with an ordinary terrestrial theodolite, to which a few simple additions have been made, it is possible to effect the determinations in a very limited space and without any corrections being necessary. In addition it will be shown that Fraunhofer's observation referred to above does not suggest a discrepancy between the best dispersive ratios as determined by theory and by practice, but rather an error which appears to have occurred in his calculations.

For the determination of the refraction ratios it is necessary to know the angle of the prism as well as the angle between the incident and emergent rays, it being understood that both are equally inclined to their refracting surfaces.

There is no difficulty in determining the *refracting angle* of a prism when it is fixed centrally and normally upon the graduated table of the theodolite, and when, by a suitable rotation of the latter, the image of a well-defined distant object externally reflected at one of the refracting faces is made to appear on the cross-wires of a fixed telescope directed towards the prism. If the refracting angle of the prism is ψ, the reflected image from the second refracting surface will appear in the telescope, when the table is rotated through plus or minus $(180^\circ - \psi)$. As the circle and table may be bound in this position and rotated as one piece back into the position of the first surface, the value of ψ can be determined from repeated observations, and in a more simple and certain manner than is possible when using the tangent telescope used by Fraunhofer for the purpose.

For the determination of the *angle of deviation* of a ray, Fraunhofer required to place the theodolite at a considerable distance from the slit through which the light entered, partly on account of the lengthening of the telescope through the extension of the eyepiece that would otherwise have been required, and partly in order to obtain only small angle corrections for the determination of which special triangulation was necessary.

It will easily be seen that both objections can be overcome, and that no angle corrections whatever are required, if the illuminated slit is placed at the focus of an achromatic objective which forms an infinitely distant image of the slit, and makes it possible to bring the measuring instrument as close as convenient to this objective.

In order to render parallel the beam of light proceeding from the heliostat mirror or other reflection apparatus through the small vertical opening, as in the arrangement of Fraunhofer, there may be used the theodolite auxiliary telescope, which for the purpose should be removed from the theodolite. For the eyepiece there should be substituted two plates having sharp straight edges forming the slit. The eyepiece tube should be adjusted until the slit is exactly in the focus of the objective. This piece of apparatus must then be so arranged that the light proceeding from the heliostat falls upon the vertical slit, and after passing through the collimator tube and objective, enters the dark room as a parallel beam. To the objective holder of this telescope, which may be fixed to the window shutter, there should be attached a small horizontal table for the support of the prism, the height being such that the light proceeding from the objective passes through the middle of the prism. The prism should be capable of rotation upon the table surface about its true vertical axis. In this way there will be formed in the room a spectrum of an infinitely distant source of light.

The problem then becomes to direct the theodolite in such a way that the angle may be determined between the image of the vertical slit when viewed directly through the objective and the various coloured images deviated through the prism. But the vertex of the angle in question is not situated at the centre of the theodolite—its opening is directed towards it. This difficulty may be easily overcome by substituting for the horizontal axis of the telescope a cylindrical axis of rotation having bearing portions of about double the original length, and by fixing it at right angles to the telescope. The theodolite may then be placed in front of the objective of the collimator, and as near as possible to the prism mounted upon it, in such a way that both telescopes are at an equal height. It should then be levelled in this position. For the determination of the angle of deviation of a particular fixed line, the telescope should be displaced sideways between the trunnion supports and the circle so rotated that the image of the fixed line appears in the telescope.

So far there is no indication as to whether or not the position of the prism is such that the incident and emergent angles of the ray under consideration are equal. As is well known, this position is found by noting the point at which the image is least deviated when the prism is rotated about its vertical axis. This involves naturally a resetting of the theodolite. When the adjustment upon the fixed line has been made in this manner and the circle has been read, the prism should be rotated through approximately 180° about its vertical axis, so that the ray is now deviated towards the opposite side. The circle should be unclamped and the telescope brought into a suitable position by displacing its axis sideways in the supports, so that, when the table carrying the correctly placed prism is rotated, the minimum deviation of the ray in question can be observed, as already described in the case of the first position. The difference between the latter and the former readings is twice the angle of deviation, 2γ (according to Fraunhofer, 2μ), that is, twice the angle between the incident and emergent rays when the angles of incidence and emergence are equal. The result thus obtained requires no further correction.

It will be self-evident that the measurement may be repeated. It then depends only upon the observer whether or not he will determine in the same way the angle of refraction of each of the other individual lines, or whether it is preferable to find by repeated measurement the angles between the individual fixed lines and by suitable summation to determine the total angle. Since for any one position of the eyepiece, all the lines do not appear simultaneously well defined, it is always necessary to measure from one of the principal fixed lines to the following one. When measuring the separation of two consecutive lines, the eyepiece must accordingly be adjusted to observe both as clearly as possible, and not into the best position for one particular line. The measurement is best made by the method first described by us, namely, by the determination in each case of the complete double angle of deviation.

If, as before, the refracting angle of the prism is ψ, and the angle between the emergent and the incident rays is denoted by γ, then, according to Fraunhofer's expression in the case under consideration, when the incident and emergent angles are equal, the index of refraction for the particular line is

$$n = \frac{\text{sine } \frac{1}{2}(\gamma + \psi)}{\text{sine } \frac{1}{2}\psi}.$$

From this expression there may be obtained, for the type of glass in question, the index of refraction of each of the Fraunhofer fixed lines A, B, C, . . . These will be indicated by the series ${}_{A}n$, ${}_{B}n$, ${}_{C}n$, . . . For another type of glass the series would be ${}_{A}n'$, ${}_{B}n'$, ${}_{C}n'$, . . .

As is well known, for the computation of an achromatic objective, it is necessary to know what is the ratio of the dispersions of the two types of glasses employed; that is to say, the value of $\frac{\Delta n'}{\Delta n}$, namely, the ratio of the small difference of n' to the corresponding difference of n.

According to the observations of Fraunhofer, this ratio is not quite constant, but varies for different parts of the spectrum. There is some doubt as to what particular ratio should be selected for the computation of an achromatic objective in order to obtain the best result. Fraunhofer decided to take the mean of the individual values of the ratio of dispersion between one fixed line and the next; that is, the mean of the values:

$$\frac{{}_{\mathrm{B}}n' - {}_{\mathrm{A}}n'}{{}_{\mathrm{B}}n - {}_{\mathrm{A}}n}, \quad \frac{{}_{\mathrm{C}}n' - {}_{\mathrm{B}}n'}{{}_{\mathrm{C}}n - {}_{\mathrm{B}}n}, \quad \frac{{}_{\mathrm{D}}n' - {}_{\mathrm{C}}n'}{{}_{\mathrm{D}}n - {}_{\mathrm{C}}n}, \quad \text{\&c.,}$$

taking into account the intensities of the corresponding rays. By practical experiment, however, he found that the most suitable value of $\frac{\Delta n'}{\Delta n}$ differed by a not inconsiderable amount from the value calculated upon the above assumption. We believe that the reason for the difference lay solely in his choice of a method of computation which did not permit of rigid checking, because the uncertainty regarding the best value of $\frac{\Delta n'}{\Delta n}$ could only be decided from the significance of this value in the formula upon which the calculation was based. In this formula, however, $\frac{\Delta n'}{\Delta n}$ involved a real differential quotient whose introduction served the purpose of finding approximately the values of n and n' for any position in the spectrum when these values were known for a particular position for which the objective satisfied the specified conditions.

Suppose, for example, it is desired to pass from one ray R, whose values $n = {}_{\mathrm{R}}n$ and $n' = {}_{\mathrm{R}}n'$ enter into the calculation, to a definite ray C, it will be evident that

$$\frac{\Delta n'}{\Delta n} = \frac{{}_{\mathrm{C}}n' - {}_{\mathrm{R}}n'}{{}_{\mathrm{C}}n - {}_{\mathrm{R}}n}.$$

In the same way, when passing to the ray D,

$$\frac{\Delta n'}{\Delta n} = \frac{{}_{\mathrm{D}}n' - {}_{\mathrm{R}}n'}{{}_{\mathrm{D}}n - {}_{\mathrm{R}}n},$$

and similarly for the ray E,

$$\frac{\Delta n'}{\Delta n} = \frac{{}_{\mathrm{E}}n' - {}_{\mathrm{R}}n'}{{}_{\mathrm{E}}n - {}_{\mathrm{R}}n}, \quad \text{\&c.}$$

In order, as far as possible, to express the conditions simultaneously for the whole of the spectrum, it is evident that the average of all such values must be chosen, taking into consideration the intensities of the various rays.

If the letter s denotes a general index for any particular part of the spectrum (corresponding to B, C, . . .), and if K denotes the corresponding intensity of the particular part, the true mean value will accordingly be

$$\text{I} \qquad \frac{\Delta n'}{\Delta n} = \frac{\int \mathrm{K} \frac{{}_{s}n' - {}_{\mathrm{R}}n'}{{}_{s}n - {}_{\mathrm{R}}n} ds}{\int \mathrm{K} ds} = \frac{p}{w},$$

where the integral is to be taken over the whole spectrum. If for the purpose of integration each expression is divided into separate portions, limited, for example, by the red end A of the spectrum and the first fixed line B, by the line B and the line C, by the lines C and D, &c., and, finally, by the last Fraunhofer line H and the violet end I of the spectrum; then with sufficient approximation the factor $\frac{{}_{s}n' - {}_{\mathrm{R}}n'}{{}_{s}n - {}_{\mathrm{R}}n}$ in each of the portions may be assumed constant and equal in each case to the arithmetical mean of the two limiting values of the particular portion. These values are obtainable from the measurements, and are therefore known. The numerator then takes the form

$$\text{II} \quad p = \tfrac{1}{2}({}_{\mathrm{R}}a + {}_{\mathrm{R}}b)\int_{\mathrm{A}}^{\mathrm{B}} \mathrm{K} ds + \tfrac{1}{2}({}_{\mathrm{R}}b + {}_{\mathrm{R}}c)\int_{\mathrm{B}}^{\mathrm{C}} \mathrm{K} ds + {}$$
$$+ \tfrac{1}{2}({}_{\mathrm{R}}c + {}_{\mathrm{R}}d)\int_{\mathrm{C}}^{\mathrm{D}} \mathrm{K} ds + \ldots + \tfrac{1}{2}({}_{\mathrm{R}}h + {}_{\mathrm{R}}i)\int_{\mathrm{H}}^{\mathrm{I}} \mathrm{K} ds,$$

where for simplification the following expressions, similar to those of Fraunhofer, have been introduced, namely:—

$$
\begin{aligned}
{}_{R}a &= \frac{{}_{A}n' - {}_{R}n'}{{}_{A}n - {}_{R}n}, \\
{}_{R}b &= \frac{{}_{B}n' - {}_{R}n'}{{}_{B}n - {}_{R}n}, \\
{}_{R}c &= \frac{{}_{C}n' - {}_{R}n'}{{}_{C}n - {}_{R}n}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
{}_{R}i &= \frac{{}_{I}n' - {}_{R}n'}{{}_{I}n - {}_{R}n}.
\end{aligned}
\tag{III}
$$

The integrals which appear in equation II, and whose sum forms the denominator of equation I, are simply expressions for the total light value of those portions of the spectrum between A and B, B and C, . . . H and I. Fraunhofer determined these values by photometric measurements of the intensities of the rays. If these are denoted by

$$ {}_{A,B}K, \quad {}_{B,C}K, \quad {}_{C,D}K \ldots, $$

then

$$ \int_{A}^{B} K\,ds = {}_{A,B}K, \quad \int_{B}^{C} K\,ds = {}_{B,C}K, \ldots \quad \int_{H}^{I} K\,ds = {}_{H,I}K, $$

and from his measurements,

$$
\begin{aligned}
{}_{A,B}K &= 0{\cdot}000 \\
{}_{B,C}K &= 0{\cdot}021 \\
{}_{C,D}K &= 0{\cdot}299 \\
{}_{D,E}K &= 1{\cdot}000 \\
{}_{E,F}K &= 0{\cdot}328 \\
{}_{F,G}K &= 0{\cdot}185 \\
{}_{G,H}K &= 0{\cdot}035 \\
{}_{H,I}K &= 0{\cdot}000
\end{aligned}
\tag{IV}
$$

Thus from equation I there is obtained approximately

$$V_{R}\left(\frac{\Delta n'}{\Delta n}\right) = \frac{\frac{1}{2}({}_{R}a + {}_{R}b)_{A,\,B}K + \frac{1}{2}({}_{R}b + {}_{R}c)_{B,\,C}K +}{{}_{A,\,B}K + {}_{B,\,C}K +} \frac{+ \frac{1}{2}({}_{R}c + {}_{R}d)_{C,\,D}K + \ldots + \frac{1}{2}({}_{R}h + {}_{R}i)_{H,\,I}K}{+ {}_{C,\,D}K + \ldots + {}_{H,\,I}K}.$$

Since the values ${}_{R}a$, ${}_{R}b$, ${}_{R}c$ . . . ${}_{R}i$, as expressed in equations III, depend not only upon the positions of the rays A, B, C . . . I, to which each refers, but also upon the ray R, to which n and n' correspond, the mean value $\frac{\Delta n'}{\Delta n}$ is necessarily also a function of R, and for this reason the suffix R is attached to it.

It will be seen that not only are the values n and n' of the refractive index dependent upon the ray R, for which in the first instance the calculation is made, but that also the mean value of the dispersion ratio $\frac{\Delta n'}{\Delta n}$ *for the same types of glass* varies according to the choice of R. Herein lies a distinct difference between the expression V introduced by us and that of Fraunhofer. For in his method only the ratios of the differences of n for *pairs of consecutive* fixed lines come into question. The mean value of $\frac{\Delta n'}{\Delta n}$, denoted by him by the letter x, is accordingly quite independent of R, that is, of the portion of the spectrum to which n and n' apply. According to Fraunhofer's method there is within the limits of the spectrum an infinite number of pairs, corresponding with n and n', but only one value of $\frac{\Delta n'}{\Delta n}$; that is, he makes this differential quotient *constant* for the whole spectrum, whereas the best value must depend, in the nature of the subject, upon the choice of the position to which it is applicable.

By computing numerically the example given by Fraunhofer, it will be shown that the difference between his results and ours is sufficiently great to warrant the assumption that the observed difference between his theory and practice had its origin solely in the inaccuracy referred to.

The calculation, in accordance with our expression ${}_{\mathrm{R}}\left(\frac{\triangle n'}{\triangle n}\right)$, must be based upon a definite position R of the spectrum. Fraunhofer has not indicated which rays correspond with the values of n and n' employed by him in the calculation of the objective from which he formed his impression regarding the best possible achromatization.

With regard to this it will be necessary to make certain assumptions. It will be assumed that the calculation is made for the two lines D and E, since these include the brightest part of the spectrum, which in the computation of an objective must be taken particularly into consideration. Our formula must first be interpreted for the case in which R coincides with one of the fixed lines, for example the D line. If, in equation V, R = D, the value of the expression cannot be determined, since the term relating to d takes the form $\frac{0}{0}$. The ambiguity can easily be removed by a rearrangement of the integral in the numerator of equation I, which in the present case must be subdivided into separate terms whose limits are A to B, B to C, C to E, E to F, F to G, G to H, H to I. The further development is performed in an entirely analogous manner, and therefore only the result is given here, namely,

$$\text{VI} \quad {}_{\mathrm{D}}\left(\frac{\triangle n'}{\triangle n}\right) = \frac{\frac{1}{2}({}_{\mathrm{D}}a + {}_{\mathrm{D}}b)\,{}_{\mathrm{A,\,B}}\mathrm{K} + \frac{1}{2}({}_{\mathrm{D}}b + {}_{\mathrm{D}}c)\,{}_{\mathrm{B,\,C}}\mathrm{K} +}{{}_{\mathrm{A,\,B}}\mathrm{K} + {}_{\mathrm{B,\,C}}\mathrm{K} +}$$

$$+ \frac{\frac{1}{2}({}_{\mathrm{D}}c + {}_{\mathrm{D}}e)\,{}_{\mathrm{C,\,D}}\mathrm{K} + \frac{1}{2}({}_{\mathrm{D}}c + {}_{\mathrm{D}}e)\,{}_{\mathrm{D,\,E}}\mathrm{K} + \frac{1}{2}({}_{\mathrm{D}}e + {}_{\mathrm{D}}f)\,{}_{\mathrm{E,\,F}}\mathrm{K} + \ldots}{+ {}_{\mathrm{C,\,D}}\mathrm{K} + {}_{\mathrm{D,\,E}}\mathrm{K} + {}_{\mathrm{E,\,F}}\mathrm{K} + \ldots}$$

From this formula it will be seen that, whenever R itself is one of the Fraunhofer fixed lines, e.g. D, the ratio ${}_{\mathrm{D}}d$, which then becomes indeterminate, must be replaced in the first term in which it occurs by the next succeeding value ${}_{\mathrm{D}}e$, and in the second by the next preceding known value ${}_{\mathrm{D}}c$.

Fraunhofer's determinations, which will be used in the calculations, are as follows for the types of glasses, Crown No. 13 and Flint No. 30:—

	n	n'
B …	1·524 312 …	1·623 570
C …	1·525 299 …	1·625 477
D …	1·527 982 …	1·630 585
E …	1·531 372 …	1·637 356
F …	1·534 337 …	1·643 466
G …	1·539 908 …	1·655 406
H …	1·544 684 …	1·666 072

${}_{B}n - {}_{D}n = -\ 3670$
${}_{C}n - {}_{D}n = -\ 2683$
${}_{E}n - {}_{D}n = +\ 3390$
${}_{F}n - {}_{D}n = +\ 6355$
${}_{G}n - {}_{D}n = +11926$
${}_{H}n - {}_{D}n = +16702$

${}_{B}n' - {}_{D}n' = -\ 7015$
${}_{C}n' - {}_{D}n' = -\ 5108$
${}_{E}n' - {}_{D}n' = +\ 6771$
${}_{F}n' - {}_{D}n' = +12881$
${}_{G}n' - {}_{D}n' = +24821$
${}_{H}n' - {}_{D}n' = +35487$

$\log\ {}_{B}n - {}_{D}n = 3\cdot 564\,7n$
${}_{C}n - {}_{D}n \ldots 3\cdot 428\,6n$
${}_{E}n - {}_{D}n \ldots 3\cdot 530\,2$
${}_{F}n - {}_{D}n \ldots 3\cdot 803\,2$
${}_{G}n - {}_{D}n \ldots 4\cdot 076\,4$
${}_{H}n - {}_{D}n \ldots 4\cdot 222\,7$

$\log\ {}_{B}n' - {}_{D}n' = 3\cdot 846\,0n$
${}_{C}n' - {}_{D}n' \ldots 3\cdot 708\,2n$
${}_{E}n' - {}_{D}n' \ldots 3\cdot 830\,7$
${}_{F}n' - {}_{D}n' \ldots 4\cdot 109\,9$
${}_{G}n' - {}_{D}n' \ldots 4\cdot 394\,9$
${}_{H}n' - {}_{D}n' \ldots 4\cdot 550\,1$

$\log\ {}_{D}b = 0\cdot 281\,3$
${}_{D}c \ldots 0\cdot 279\,6$
${}_{D}e \ldots 0\cdot 300\,5$
${}_{D}f \ldots 0\cdot 306\,7$
${}_{D}g \ldots 0\cdot 318\,5$
${}_{D}h \ldots 0\cdot 327\,4$

${}_{D}b = 1\cdot 911$
${}_{D}c = 1\cdot 904$
${}_{D}e = 1\cdot 997$
${}_{D}f = 2\cdot 026$
${}_{D}g = 2\cdot 082$
${}_{D}h = 2\cdot 125$

$\frac{1}{2}({}_{D}b + {}_{D}c) = 1\cdot 907$
$\frac{1}{2}({}_{D}c + {}_{D}e) = 1\cdot 950$
$\frac{1}{2}({}_{D}e + {}_{D}f) = 2\cdot 011$
$\frac{1}{2}({}_{D}f + {}_{D}g) = 2\cdot 054$
$\frac{1}{2}({}_{D}g + {}_{D}h) = 2\cdot 104$

$\log\ \frac{1}{2}({}_{D}b + {}_{D}c) = 0\cdot 280\,4$
$\frac{1}{2}({}_{D}c + {}_{D}e) \ldots 0\cdot 290\,0$
$\frac{1}{2}({}_{D}c + {}_{D}e) \ldots 0\cdot 290\,0$
$\frac{1}{2}({}_{D}e + {}_{D}f) \ldots 0\cdot 303\,4$
$\frac{1}{2}({}_{D}f + {}_{D}g) \ldots 0\cdot 312\,6$
$\frac{1}{2}({}_{D}g + {}_{D}h) \ldots 0\cdot 323\,0$

$\log\ {}_{B,C}K = 8\cdot 322\,2$
${}_{C,D}K \ldots 9\cdot 475\,7$
${}_{D,E}K \ldots 0\cdot 000\,0$
${}_{E,F}K \ldots 9\cdot 515\,9$
${}_{F,G}K \ldots 9\cdot 267\,2$
${}_{G,H}K \ldots 8\cdot 544\,1$

	Log.	Number.
$\frac{1}{2}({}_{D}b + {}_{D}c) \cdot {}_{B,C}K$	8·6026	0·040
$\frac{1}{2}({}_{D}c + {}_{D}e) \cdot {}_{C,D}K$	9·7657	0·583
$\frac{1}{2}({}_{D}c + {}_{D}e) \cdot {}_{D,E}K$	0·2900	1·950
$\frac{1}{2}({}_{D}e + {}_{D}f) \cdot {}_{E,F}K$	9·8193	0·660
$\frac{1}{2}({}_{D}f + {}_{D}g) \cdot {}_{F,G}K$	9·5798	0·380
$\frac{1}{2}({}_{D}g + {}_{D}h) \cdot {}_{G,H}K$	8·8671	0·074

Numerator of ${}_{D}\left(\frac{\Delta n'}{\Delta n}\right)$ . . . . 3·687 log = 0·5666

Denominator $= {}_{A,B}K + {}_{B,C}K + {}_{C,D}K + \ldots + {}_{H,I}K = 1·868$ log = 0·2714

log ${}_{D}\left(\frac{\Delta n'}{\Delta n}\right)$ . . . . 0·2952

$${}_{D}\left(\frac{\Delta n'}{\Delta n}\right) = 1·973.$$

Proceeding in a similar manner with the calculation for the fixed line E, (R = E), it will be found that ${}_{E}\left(\frac{\Delta n'}{\Delta n}\right) = 2·028.$

The value 1·98 found by Fraunhofer to be the most suitable in practice falls between the two values determined by us. There is, therefore, no ground for basing upon the Fraunhofer observations any doubt as to the correspondence between experience and correct theory.

Finally, it should be observed that the choice of R, that is, of n and n', upon which the calculations are based, has been left arbitrary within the limits of the spectrum, because our formulæ V and VI give the best value of $\frac{\Delta n'}{\Delta n}$ for any value of R. Since, however, in every achromatic combination the most intense rays are always the most important, it would be unsuitable to choose R too near one end of the spectrum. It is always preferable to proceed either from one of the two fixed lines D and E, or from one lying between them.

APPENDIX II

Trigonometrical Formulæ for the General Case of the Refraction of Light by a System of Centred Spherical Surfaces

BY

DR. L. VON SEIDEL

[Reprinted from the *Proceedings of the Mathematical Physical Class of the Imperial Bavarian Academy of Science*, dated 10th November, 1866. The notation has been altered to correspond with that used throughout this book.]

The mathematical formulæ, by means of which the path of a ray of light through a system of centred spherical surfaces can be accurately traced trigonometrically, and which are suitable for the use of the practical optician who wishes to perfect an already approximately determined design, are now generally well known. These formulæ, however, do not deal with the geometrical problem in its most general form, but are limited to the consideration of such rays of light as originally lie in a plane containing the optical axis and which consequently lie in the same fixed plane after each successive refraction. They are applicable to all the rays in the cone of light proceeding only from a point which lies in the centre of the field of view, that is, in the prolongation of the optical axis. In all other cases it is solely the rays lying in the plane which cuts the cone in two halves, that is, the plane through the apex of the cone and the optical axis, that satisfy the particular conditions of the formulæ.

If, for example, this axis is supposed to be directed horizontally, and the luminous point to lie anywhere directly above it, then it is not possible by any of the trigonometrical formulæ hitherto produced to trace rays which proceed from the point towards either the right or the left hand of the entrance surface of the system. It is only possible to trace those rays which are incident exactly on the line of separation of the two halves of this surface. Hitherto it has been necessary to assume these latter rays to be representative of the whole cone of light.

Before the introduction of the method of computation here described, which was subsequently used in the Steinheil factory, the path of any other ray even through a simple lens could only be accurately calculated with difficulty by means of the calculus. It is true that the Fraunhofer telescope objective satisfied requirements of the most severe kind even with respect to rays lying out of the axial plane. But since the objective had in any case to satisfy the same equations in order that the representative rays in the plane of the axis referred to above might be brought to the best possible focus, and since after satisfying this condition in a simple doublet objective nothing more remains to be disposed of, the extension of the computation to include the side rays was not a necessary requirement for the determination of the objective.

Even Bessel's masterly theoretical discussion concerning the qualities of the Königsberg Observatory heliometer objective did not take into account the rays lying out of the plane of the axis.* On the other hand, Gauss, in the development of his approximate formulæ (*Dioptrische Untersuchungen*), proceeds from equations which are rigidly general, but he gives no directions for the computation of the three angles: Θ, λ, λ', which enter into his equations, since the object of his investigation, which, under the conditions specified, was sufficient, was to establish that the cosine of the first and the sines of the two last differed from unity only by small amounts of the second order.

In a similar manner, my investigations, described in the *Astronomische Nachrichten*, No. 1027, &c., the object of which was to develop for the most general case terms of the order of the so-called spherical aberrations, were based only upon approximate values of the trigonometrical functions of the angles.

As was stated also by Bessel in the reference previously mentioned it is to be understood that the evolution of rigid formulæ by means of which, for any position of the incident rays, the corresponding positions of the refracted rays may be determined, offers no real mathematical difficulty. Unless the choice of the values which serve to determine these positions is wholly suitable, the computation formulæ may easily be obtained in a form from the actual use of which even an experienced computer might shrink, especially since the tracing of individual rays in three dimensions is involved, when a considerable number of refractions have to be considered. The formulæ may be obtained, for example, in such a form that at each individual deviation which the ray suffers, either an awkward spherical triangle has to be solved, or the results obtained by successive approximations. Since the increasing demands as regards aperture and angular field, for example, in the case of photographic objectives, make it no longer possible to ignore the rays lying out of the axial plane, it is hoped that the following method of computation, which has already been frequently submitted to the test of actual use, will prove of service to interested opticians.

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* *Astronomische Untersuchungen*, Vol. I, Part II, par. 18.

The immediate inducement to summarize definitely this method arises from the wish expressed by Dr. Steinheil. In the optical works established by him my formulæ have been frequently used during the past year by his son, Dr. Adolf Steinheil, who has found that the labour of reckoning a ray lying out of the axial plane is only slightly greater than that involved in the computation of two rays in the axial plane, similar care in both cases being taken to avoid errors in calculation.

The formulæ accordingly would appear to offer, as nearly as possible, the greatest attainable ease of computation, since not only in the nature of the subject are two projections required in all cases for the determination of the positions in space, where in the axial plane one is sufficient, but since also the general case is complicated, owing to each individual value being affected by errors arising from a considerable number of variables independent of one another.

When calculating an optical system which has to be constructed, it is necessary, in order to obtain the requisite accuracy in the tracing of each ray, to perform the whole computation twice independently, in accordance with the equations which have been evolved and which are sufficient for the determination of the desired quantities, or in addition to these equations to use special checking formulæ for the verification of the numerical values obtained. The latter method of checking, assuming naturally that the checking formulæ deal exhaustively with the individual stages of the computation, is the more advantageous when one computer has to do the entire work, because as is well known in such a case the same errors may be very easily repeated at the same place. In the evolution of the following scheme of computation, particular attention has, therefore, been paid to the production of suitable checking formulæ.

The principle upon which the computer may determine which portions of the numerical calculation have been verified by satisfying correctly a particular checking formula, and which not, is essentially simple. If, for the calculation of a number of unknown quantities, an equal number of equations has been obtained, then the mathematical relationship of each of the unknown quantities to the known ones is definitely established. These variables, or any of them, cannot satisfy any further supplementary condition that does not itself follow from its corresponding mathematical function, which is already fixed.

Each supplementary equation or check formula for the unknown quantities must, therefore, be derived as an identity from some, or a combination of all, of the equations already used for the determination of these unknown quantities. In the latter instance, which quite frequently arises in the case of quantities of special importance, it is only necessary to consider which of the fundamental equations for the unknown quantities are necessary and sufficient for the derivation from them of the supplementary or check equations. It will be clear that the numerical fulfilment of a check equation is a test only of the correct fulfilment of the particular fundamental equation from which it has been mathematically derived, and not of the other equations to which it has no relation.

The rectilinear ray which is refracted by one of the spherical surfaces of a centred optical system may meet this surface in a point, P. Suppose there is placed at the centre, C, of the spherical surface a plane normal to the optical axis, then the incident ray produced if necessary in either direction will inter-

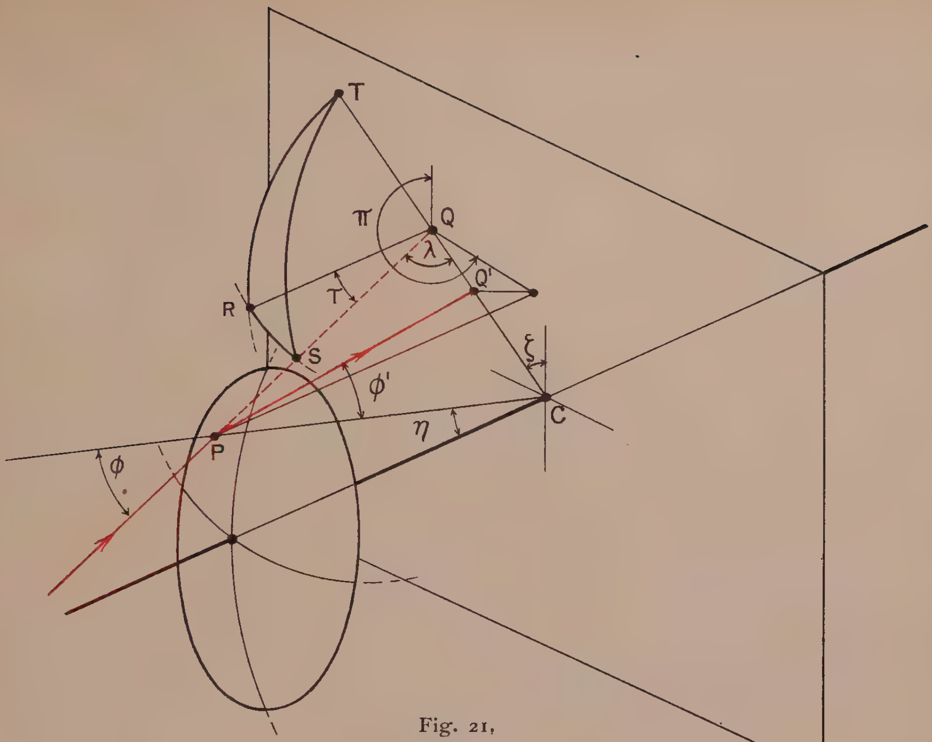

Fig. 21.

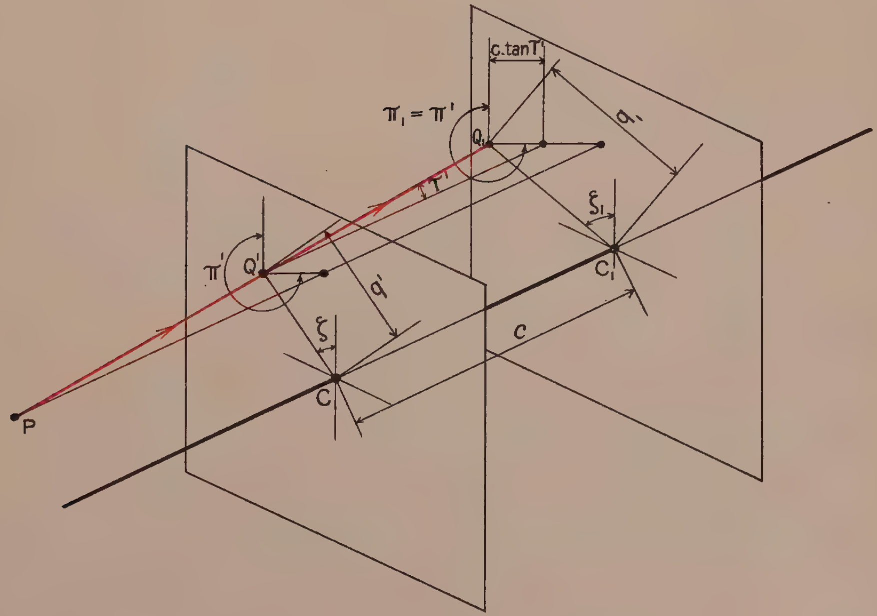

Fig. 22.

[*Facing p. 182.*

sect this plane in Q, whereas the refracted ray will intersect it at Q′. The position of either ray will be completely determined in space by four elements which may be chosen in various different ways. (See figs. 21 and 22.)

For this purpose there will be selected two co-ordinates that will determine the positions of the points Q or Q′ respectively in the transverse plane passing through the point C, and two angles that will define the directions of the rays, or their virtual prolongations as they pass through the points Q and Q′ respectively. The problem then is to calculate from the given four elements for the incident ray the corresponding four elements for the refracted ray, under the assumption that the refractive index of the glass and the curvature of the refracting surface are known.

Since, further, when there are several consecutive refractions, in passing from one to the other, the positions of the point C, and therefore also the positions of the transverse planes, are different in each case, the relationship between the co-ordinates in one plane and those in the succeeding plane must be determined.

The plane of the triangle PQC contains the incident ray PQ and the normal to the surface, PC. According to the law of refraction, it also contains the refracted ray PQ′; or, in other words, the two planes PQC and PQ′C coincide. They have in consequence a common line of intersection with the transverse plane passing through C; that is, the three points Q, Q′, and C, lie in one straight line, or the two radii vectors drawn from C to Q and Q′ have one and the same direction.*

If polar co-ordinates are used for the determination of the positions of Q and Q′ in the transverse plane, that is, the above-mentioned radii vectors CQ = q and CQ′ = q', in conjunction with the angle which these radii vectors make with a fixed direction, the advantage is obtained that the angular co-ordinates do not alter with the refraction.

The fixed direction from which the polar angles are measured may be chosen at will, but it must be the same throughout the whole series of transverse planes that come into consideration;

---

* When the spherical surface is a reflecting instead of a refracting surface, these two directions will be diametrically opposed. This special, and at the same time particularly simple case, will not be dealt with further.

that is, in these various planes the co-ordinates must be referred to directions that are parallel to one another, and which are drawn in the same sense from one side of the optical axis. For convenience of expression, it will be assumed that the direction is *upwards* from the point C, the optical axis being regarded as horizontal. The angles included between this direction and the various radii vectors drawn from C (and never produced backwards through C), are reckoned from 0° to 360°,* all in one fixed sense which will be assumed to be clockwise. When reckoned in this way, ζ denotes the common inclination of the two radii vectors CQ and CQ′ to the upward direction, while q and q' represent the lengths of these radii vectors, which are necessarily positive.

By means of the polar co-ordinates q, ζ, and q', ζ', the points of intersection Q and Q′ respectively of the ray before and after refraction with the fixed transverse plane are defined. The angles that are still required for the determination of the direction of the ray which reaches one or other of these points, can be briefly described in the language of spherical astronomy as the apparent distance τ from the centre of the field of view of that point in the heavens to which the ray, when produced backwards, points, and as the angle of orientation π, of the same point in the heavens, with reference to the centre of the field of view and reckoned from the upward direction.

Independently of these technical expressions, the same values may be defined, as follows:—Suppose from the point Q there is drawn a line parallel to the optical axis towards the side from which the light originally comes. This line will then make with that portion of the incident ray or its prolongation on the same side of the transverse plane an angle which will be denoted by τ (fig. 21). In the more important cases this angle will be small. Further, if the portion of the ray mentioned above is projected upon the transverse plane passing through C, then there is included between its projection and the vertical direction from Q upwards, an angle π. This angle will be reckoned in a clockwise direction through 360° from the vertical upwards direction, exactly as in the previous case of the angle ζ.* For the refracted ray in place of τ and π there are two analogous angles τ' and π', whose vertexes lie at Q′.

---

* It is, moreover, permissible to subtract 360° from the angle calculated. For example, instead of an angle in the third or fourth quadrants, negative obtuse or acute angles may be introduced, and as many multiples as may be desired of one rotation may be added or subtracted, because thereby neither the directions determined by the angle nor the trigonometrical functions of the angle are altered. Halved or otherwise subdivided angles do not arise in the expressions.

In order to pass from the values q, τ, π, to the new values q', τ', π', after refraction in a manner convenient for computation, it is necessary to introduce certain auxiliary angles. The interior angle at Q in the triangle PQC will be denoted by λ (fig. 21), and the analogous angle at Q′ in the triangle PQ′C by λ'. Further, the angle at P in the first triangle, that is, the angle of incidence of the ray, will be denoted by ϕ. The corresponding angle in the second triangle, that is, the angle of refraction, will be denoted by ϕ'.

The ratio of the sines of these two latter angles is, in accordance with the law of refraction, a constant for any particular colour, depending upon the nature of the two media; thus

$$\frac{\sin\phi}{\sin\phi'} = \frac{n'}{n}.$$

It will be assumed that the radius of the refracting spherical surface, expressed in the same units of length as the radii vectors q and q', is denoted by r.

These values will be reckoned always as positive, and the two cases in which the convexity or the concavity of the surface is turned towards the direction from which the light originally came will be distinguished from one another by ± signs. Throughout the following discussion, where such a ± sign appears, the upper sign refers to the former case and the lower to the latter. Both might be included in one and the same formula if the radius r, as is the common practice, were distinguished by an appropriate sign, and at the same time the definition of the angles λ and λ' were slightly modified. For the purpose of numerical computation, and particularly when it is desired to represent the results geometrically, the arrangement adopted here would appear to be somewhat more convenient.†

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* The values τ and π can be defined analogously to those of q and ζ; thus, τ may be defined as the *angular* value of the radius vector, and π as the polar angle of the point in which the ray intersects a plane normal to the axis, the ray being supposed to lie on the side from which the light from an infinite distance proceeds. (See the additional note at the end of this Appendix.)

† The double-sign convention, which was used throughout the original work, has not been adopted in the translation, although it has been retained in this Appendix.—Ed.

Corresponding with the given values of τ, π, and ξ, the value of λ must first be calculated. Suppose that about the point Q (fig. 21) in the transverse plane there is described, on that side of the plane from which the light originally proceeded, a hemisphere having any chosen radius. For the moment the point of intersection with the surface of the sphere, of a line drawn through Q parallel to the optical axis, will be denoted by R; the point of intersection of the ray PQ or its prolongation will be denoted by S; and the point in the transverse plane where the radius vector CQ, produced beyond Q, intersects the surface of the hemisphere will be denoted by T. In the spherical triangle RST, the side RT = 90°, the side RS = τ, and the angle at R has either the value $\xi - \pi$, or $\pi - \xi$. The side ST is equal to the exterior or the interior angle at Q in the plane triangle PQC, namely, either $180° - \lambda$, or λ, according as the point P lies on the same side of the transverse plane as the supposed hemisphere, or upon the opposite side; that is, according as the convexity of the refracting surface is directed towards the side from which the light originally proceeds or towards the opposite side.

Adopting the double-sign convention already described, there is obtained the formula:

$$\text{(1)} \qquad \cos\lambda = \mp \sin\tau \cos(\pi - \xi).$$

The determination of λ from its cosine is free from ambiguity, since the angle, according to its definition, must lie between 0° and 180°. At the same time it is suitable for numerical calculation, since in the more important cases that arise, λ differs but little from 90°.

In the plane triangle PQC, there are now known the sides PC = r and QC = q, and the angle λ at Q. It is therefore possible to calculate the angle at P, that is ϕ, the angle of incidence of the ray. Thus:

$$\text{(2)} \qquad \sin\phi = \frac{q \sin\lambda}{r}.$$

From this value there may be obtained the analogous angle in the triangle PQ′C, namely, the angle of refraction:

$$\text{(3)} \qquad \sin\phi' = \frac{n}{n'} \sin\phi.$$

In the latter triangle the angle at Q′, that is λ', is now also known, since the third angle at C is common to the two triangles, and therefore $\lambda' + \phi' = \lambda + \phi$. Thus:

$$(4) \qquad \lambda' = \lambda + (\phi - \phi').$$

Further, the side CQ′ $= q'$ of the same triangle may be found from the formula:

$$(5) \qquad q' = \frac{r \sin \phi'}{\sin \lambda'} = q \frac{n}{n'} \cdot \frac{\sin \lambda}{\sin \lambda'}.$$

By calculating this value from both equations, which must give results that correspond exactly, the correct determination of $\sin \phi$ and $\sin \phi'$ from equations (2) and (3) is checked at the same time. Apart from this, the quotient $\frac{\sin \lambda}{\sin \lambda'}$ will be required later.

The point Q′ is now known from its polar co-ordinates q' and ζ. There are still unknown the values τ' and π' for the refracted ray. One of the necessary equations is obtained directly from equation (1), when the symbols λ, τ, and π are accented, ζ being retained unaltered, since naturally in the case of the refracted ray there exists a spherical triangle that is strictly analogous to the previous one. The second equation is obtained most directly from the consideration that, according to the convention previously used, the angle at T in the first triangle must coincide with the corresponding angle of the second. For since the plane of the great circle ST passes through the incident ray and through the centre C it therefore remains unaltered, in accordance with the law of refraction, when the refracted ray is considered in place of the incident ray. The plane of the great circle RT, however, which is inclined to the former at an angle T, contains the two straight lines QT, which, when produced beyond Q, intersects the axis in C, and QR, which is parallel to the axis. It contains, therefore, the axis itself, and is the plane containing the axis and the radius vector CQ. Since now CQ′ and CQ coincide in direction, this plane also remains unaltered when Q′ is substituted for Q, and the statement made above is accordingly justified.

The expression for sin T, as obtained from the first spherical triangle, namely $\frac{\sin \tau \sin (\pi - \zeta)}{\sin \lambda}$, must accordingly be equal to the similar expression obtained from the second triangle, namely, that in which τ', π', λ' appear instead of τ, π, λ. For the determination of τ', π', there are now available two equations:

$$(6) \quad \sin\tau' \sin(\pi' - \zeta) = \frac{\sin\lambda'}{\sin\lambda} \cdot \sin\tau \sin(\pi - \zeta),$$

and

$$(7) \qquad \sin\tau' \cos(\pi' - \zeta) = \mp \cos\lambda'.$$

Since τ' represents an acute angle whose sine is necessarily positive, the signs of the sine and cosine of $(\pi' - \zeta)$ are known, and there can therefore be no doubt as to the quadrant in which the angle lies. As regards τ', it should be remarked that the angle itself is not required further in the calculation. It appears only as the sine or the tangent. The latter can be calculated directly, for if the value of cos T, instead of sin T, is determined from the two spherical triangles, its value, as derived from the first triangle, equals $\frac{\cos\tau}{\sin\lambda}$, and from the second, $\frac{\cos\tau'}{\sin\lambda'}$. Therefore

$$\cos\tau' = \frac{\sin\lambda'}{\sin\lambda}\cos\tau.$$

As compared with equation (6), this equation has the advantage of greater simplicity. For numerical computation in the cases of most practical importance, however, in which τ and τ' are small, it permits of a less accurate determination of the angle. When it is combined, however, with equation (6), there is obtained equation:

$$(8) \qquad \tan\tau' = \tan\tau \frac{\sin(\pi - \zeta)}{\sin(\pi' - \zeta)},$$

so that the computer has the choice of obtaining the logarithm of the tangent, which will be used in any case in the continuation of the computations to the succeeding refraction, either by the use of tables from the value of the angle deduced from the sine previously found, or independently from quantities that are already known. For the checking of the calculations the agreement of these two values has, however, little value, since equation (8) is derived directly, with the aid of the equation for cos τ', from equation (6), which would be used in any case. This latter equation, in which values that are approximately equal to unity only occur, provides in no way a sensitive test.

A much better check is obtained from the following equation:—

$$\text{(I)} \qquad \pm\frac{\sin(\phi - \phi')}{\sin(\pi - \pi')} = \frac{\sin\lambda \sin\tau'}{\sin(\pi - \zeta)} = \frac{\sin\lambda' \sin\tau}{\sin(\pi' - \zeta)},$$

provided that not only the numerical values of the two last expressions, which are identical with equation (6), but also that of the first is taken. The truth of these equations may be verified by multiplying the numerator and denominator of the left-hand term by $\sin\tau \sin\tau'$, by substituting $(\lambda' - \lambda)$ for $(\phi - \phi')$ in the numerator in accordance with equation (4), by expanding the denominator $\sin(\pi - \pi') = \sin[(\pi - \zeta) - (\pi' - \zeta)]$, and by substituting for each of the products, $\sin\tau \cos(\pi - \zeta)$, $\sin\tau' \cos(\pi' - \zeta)$, and $\sin\tau' \sin(\pi - \zeta)$, their values derived from equations (1), (7), and (6). From this it will be evident at the same time that, provided the substituted numerical values of $\sin(\phi - \phi')$ and $\sin(\lambda' - \lambda)$ agree, the true solution of equation (I) arises directly from equations (1), (6), and (7), and does not serve as a check for any but these equations. On the other hand, on the understanding that the constants r and $\frac{n}{n'}$ are not incorrect, the values of both $\sin\phi$ and $\sin\phi'$, together with that of q', may be checked from the double determination of this last value, by means of equation (5). It is still possible, however, that either ϕ or ϕ' may be wrongly derived from the correct sine, and thereby, or through a mistake in the subtraction itself, an incorrect value of $\phi - \phi'$, and consequently of λ', might be obtained without the error being detected by the checks hitherto mentioned.* It is desirable, therefore, to have a further check on the transformation in question. The most convenient would appear to be derived as follows:—

Since
$$\frac{\sin\phi}{\sin\phi'} - \frac{\sin\phi'}{\sin\phi} = \frac{n'}{n} - \frac{n}{n'},$$

substituting $\frac{n}{n'} = \tan\omega$ in this equation, there is obtained the equation:

$$\frac{(\sin\phi)^2 - (\sin\phi')^2}{\sin\phi \sin\phi'} = \cot\omega - \tan\omega,$$

---

* If $\phi - \phi'$ were correct and λ' only were in error, then test equation (I) would disclose the error; similarly, if from correct values of $\cos\lambda$ and $\sin\lambda$ a false value of λ were derived, the error of the latter would be communicated to λ'.

and hence

$$\text{(II)} \qquad \frac{\sin(\phi - \phi')\sin(\phi + \phi')}{\sin\phi \sin\phi'} = 2\cot 2\omega.\text{*}$$

The quantity on the right-hand side is constant for all rays of the same colour which are refracted between the same two media. If the computation has to be made simultaneously for several such rays, it is not necessary actually to derive the constants, but only to observe whether or not the left-hand terms have one and the same value for these different rays.

Those who rarely make mistakes, and who need not consider the danger of having later to repeat a large part of their calculations, can generally dispense with the determination of ω and cot 2 ω, so far as the ordinary case is concerned in which the ray passes through two consecutive surfaces from a medium A into a medium B and thence again into the medium A, as, for example, from air into a glass lens and out again into air. In this case the values of $2 \cot 2\omega = \frac{n'}{n} - \frac{n}{n'}$ are opposite to one another at the two consecutive refractions. It suffices accordingly to observe whether the left-hand term of equation (II) assumes opposite numerical values.

On the other hand, if the constant 2 cot 2ω has been calculated, the determination of $\sin(\phi - \phi')$ for equations (I) and (II) may be avoided by substituting in equation (I) this value as obtained from equation (II), thus combining the two checks. Most computers will endeavour to arrange that the verification will be evident at the earliest possible stage.†

In special cases, as for example when determining the aperture of any refracting surface, it may be of interest to know the value of the (acute) angle η between the normal PC (fig. 21) and the axis, which otherwise, according to the scheme of computation described, would not be required. It may be obtained most conveniently from the consideration that the distance of the point P from the transverse plane passing through the centre C may be measured on the one hand by means of $r \cos \eta$, and on the other hand by $PQ \cos \tau$. Accordingly $\cos \eta = \frac{PQ}{r} \cos \tau$. If, for the ratio of the two sides of the triangle PCQ, namely, $\frac{PQ}{r}$, the ratio of the sines of their opposite angles is substituted, there will be obtained the equation:

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* The numerator of the left side of this equation is rigidly correct only for small values of ϕ. The true expression $(\sin\phi - \sin\phi')(\sin\phi + \sin\phi')$ would, however, be of no value as a check on the derivation of the angles ϕ and ϕ' from their sines. In any case, as explained later, the absolute value of the expression is of secondary importance for checking purposes.—Ed.

† In no case can equation (II) make unnecessary the check which is obtained for sin ϕ from the twofold calculation of q' in equation (5), since the former is in no way derived from sin ϕ.

$$\cos \eta = \frac{\sin (\lambda + \phi)}{\sin \lambda} \cos \tau = \frac{\sin (\lambda' + \phi')}{\sin \lambda'} \cos \tau'.$$

The identity of these two expressions, which is a necessary consequence of the circumstance that the normal PC occupies the same position for both the refracted and the incident rays, may be seen directly from equation (4) combined with that which has already been used in obtaining equation (8). The determination of a small angle η from its cosine is unsatisfactory where numerical precision is desired. It is not easy, however, to arrange for an accurate determination of this angle, and accordingly, since the angle does not reappear here, the derivation of one of the less elegant formulæ which permit of a more precise determination will be dispensed with.

The equations so far given contain everything that relates to the refraction at a single surface. If the refracted ray meets a second refracting surface, then in passing to the latter, τ' and π' become the new values of τ and π. If letters with the suffixes 1, 2, . . . are used to distinguish the values of the second, third, &c., refractions which correspond with those of the first surface denoted by similar letters without suffixes, it follows that

$$(9) \qquad \tau_1 = \tau' \text{ and } \pi_1 = \pi',$$

also $n_1 = n'$. On the contrary, q_1 and ζ_1 are not identical with q' and ζ', which is equal to ζ, since these latter co-ordinates are reckoned in the transverse plane through the centre C of the *first* refracting surface to which both Q′ and Q correspond (fig. 22), whereas the point Q_1, at which the once refracted ray intersects the transverse plane through the centre C_1, refers to the *second* refracting surface. The distance of the latter plane from the former, that is, of the point C_1 from C, expressed in the same units as r and also as q and q', is here denoted by the symbol c. This value is positive when C lies on the side of C_1 from which the light originally came, and negative in the opposite case. If it is desired to introduce the axial thickness d of the medium between the two refracting surfaces, there may be used the expression $c = d \mp r \pm r_1$, where, before each of the radii r and r_1 the upper or the lower sign is to be used, according to the nature of the surface to which it corresponds, in conformity with the general rule.

The most convenient method of reckoning the values q_1 and ζ_1 is obtained most readily by thinking of the point Q′, and the portion of the refracted ray lying between it and Q_1 as being projected in a direction parallel to the axis upon the transverse plane containing the new centre C_1 (fig. 22). The projected length of this portion is $c \tan \tau'$, the sign being neglected. If, in this plane the abscissæ are considered as being measured parallel to the line and the ordinates at right angles to it, then the difference of the abscissæ of the ends of the portion of the line in question is $c \tan \tau'$, while both ordinates are equal. From these considerations there are obtained the following equations for the determination of q_1 and ζ_1:

$$\text{(10)} \qquad q_1 \sin(\pi' - \zeta_1) = q' \sin(\pi' - \zeta),$$

and

$$q_1 \cos(\pi' - \zeta_1) = q' \cos(\pi' - \zeta) - c \tan \tau'.$$

Since the first value is positive, the quadrant of $(\pi' - \zeta_1)$ is fixed. For purposes of checking there may be used the equation (III), derived from the above formulæ, namely,

$$\text{(III)} \qquad \frac{c \tan \tau'}{\sin(\zeta - \zeta_1)} = \frac{q_1}{\sin(\pi' - \zeta)} = \frac{q'}{\sin(\pi' - \zeta_1)}.^{*}$$

Having found q_1 and ζ_1, and since now τ_1, π_1, q_1, and ζ_1 are known, there should be repeated with respect to the second refracting surface all the calculations that were made with

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* When equations (III) and (8) are combined there is immediately obtained the equation:

$$\frac{c}{\sin(\zeta - \zeta_1)} = \frac{q_1}{\tan \tau \sin(\pi - \zeta)} = \frac{q'}{\tan \tau' \sin(\pi' - \zeta_1)}.$$

Of these equations, the first may be used instead of (III) after the calculation of q_1 and ζ_1, and the other after the calculation of $\tau_2 = \tau_1'$ and $\pi_2 = \pi_1'$, from equations (6) and (7).

respect to the first, in accordance with equations (1) to (7). There are thus obtained in their order the values of λ_1, ϕ_1, ϕ_1' (with the aid of $n_1 = n'$ and $n_1' = n_2$), and λ_1', q_1', $\tau_1' = \tau_2$, $\pi_1' = \pi_2$. Further refractions as they arise may be dealt with in precisely the same manner.

Lastly, it remains to investigate where the final refracted ray intersects the plane in which the image is considered to be. For this purpose the formulæ (10) may be used if for τ' and π' the last determined values of these quantities are introduced, and by q' and ζ are understood the polar co-ordinates of the point in which the ray in its final position intersects the transverse plane through the last centre, and if also c represents the distance of the image plane from this last centre. This distance is reckoned positive in the case when the rays proceeding in their original direction arrive at the centre plane before the image plane. The quantities q_1 and ζ_1 then become the polar co-ordinates of the point where the image plane is intersected by the emergent ray. As in all cases, the radius vector is reckoned outwards from the axis, and the polar angle is reckoned in a right-handed direction from a vertical drawn upwards.

Equations (10) may also be used when, although τ and π in the initial position of the incident ray are given, its point of intersection with the plane through the first centre is not indicated directly, but instead there is given the point in which it intersects some other plane normal to the axis, such as, for example, the plane of the circle which forms the periphery of the first spherical surface or the plane of an object under observation. If the polar co-ordinates in such a plane are denoted by z and ξ, and c^* is the separation of this plane from that containing the first centre C, and, if the last-mentioned value is reckoned positive in the case which in practice is most common, namely, from C outwards towards the side from which the rays originally proceeded, then the known quantities τ, π, z, ξ, c^*, and the unknown quantities q and ζ, play exactly the same part as the quantities τ', π', q', ζ, c, q_1, ζ_1, respectively, of the preceding discussion.

Thus there may be obtained the equations:

$$(11) \qquad \begin{aligned} q \sin(\pi - \zeta) &= z \sin(\pi - \xi), \\ q \cos(\pi - \zeta) &= z \cos(\pi - \xi) - c^* \tan \tau, \end{aligned}$$

and the check formula:

$$\text{(IV)} \qquad \frac{c^* \tan \tau}{\sin(\xi - \zeta)} = \frac{z}{\sin(\pi - \zeta)} = \frac{q}{\sin(\pi - \xi)}.$$

In the actual use of the formulæ it will be evident to the computer that very many of the quantities that occur appear in an entirely similar manner in several of the equations, and that accordingly the labour is considerably reduced. If, for instance, the cosine and sine of $(\pi - \zeta)$ have been found from the last equations, the value of the cosine can again be used directly in equation (1), and that of the sine in equations (6) and (8). Similarly, the sine and cosine of $(\pi' - \zeta)$, determined from equations (6) and (7), appear again in (10) and the sine in (8); the difference $(\phi - \phi')$ in equation (4), and its sine in the check equations (I) and (II); the ratio $\frac{\sin \lambda}{\sin \lambda'}$ appears in (5), and its reciprocal in (6), as well as in the equation for $\cos \tau'$; and so on.

There is one exception, namely, the case of a *plane* refracting surface, to which the method described for the determination of the position of a ray is not applicable. In this instance the corresponding centre plane lies at infinity, and it cannot therefore be made use of. A remedy at once offers itself by assuming the transverse plane to coincide with the refracting plane itself, and accordingly for q and ζ to choose the polar co-ordinates of the point in which the plane surface is intersected by the ray. If these have not been previously given, they may be determined from the equations (11) when no previous refraction has occurred, and in other cases from equations (10). Naturally for c^* or c there must now be taken that value which represents the distance from the preceding transverse plane to the refracting plane itself. By the refraction at this plane neither q nor ζ is altered, since the point of incidence corresponds also with that of the refracted ray. It follows that $q' = q$.

The other equations of refraction may also be simplified, since, the normal being parallel to the axis, $\phi = \tau$, and $\phi' = \tau'$. It will readily be seen that $\sin \tau' = \frac{n}{n'} \sin \tau$, and also $\pi' = \pi$, since the plane of refraction to which the ray is confined itself contains in this instance the straight line drawn parallel to the axis through the point of incidence. The four elements,

τ', $\pi' = \pi$, $q' = q$, ζ, which determine the position of the refracted ray are accordingly known. If there is occasion to trace the path still further, and for this purpose to introduce a new transverse plane, the equations (10) may again be used, on the understanding that the point at one end of the distance c lies in the plane refracting surface itself.

The above scheme of computation, which obviously can also be applied to the special case of a ray lying in the axial plane, corresponds very closely in form with that which is in general use for such a case.

In conclusion, it must be remarked that there is a somewhat different method which is particularly suitable under certain conditions for tracing through an optical instrument the path of the light, either in or out of the axial plane. The method consists in finding directly, not the actual values corresponding with the path of a ray after any desired number of refractions, but only their deviations from values that have been determined from the approximate formulæ of the first order. According to this method it is necessary only to deal with small quantities which can be specified accurately enough with few significant figures, since they represent directly what appear in the optical image as aberrations. This treatment of the subject lends itself to the use of elegant formulæ, which stand in a wholly analogous relationship to those expressing the errors of the third order that have already been evolved by me for the most general case of three dimensions, as the "equations of finite differences" do to the differential formulæ. This method, nevertheless, differs rather too much from the customary computing method of the optician, whose practical requirements have been kept particularly in view in the present publication. Fuller details, therefore, will be reserved for another occasion.

Authors' Additional Note

Since the quantities π and τ determine the *direction* of the ray, other quantities which depend only upon the direction may be derived from these values alone. To these, for example, belongs the angle which the ray makes with the fundamental plane containing the axis, and which is supposed to lie vertically. It is the experience of Dr. A. Steinheil in the practical use of the formulæ that a knowledge of this angle is useful, and it will accordingly be denoted here by the symbol k.

Suppose there is described, on a sphere drawn with any radius about a point on the axis, the great circle in which the vertical fundamental plane intersects the surface of the sphere. On this circle let I represent the point to which the axis is directed; and, further, let the point II be known by its spherical co-ordinates π and τ. Then the arc II, III, from the point II outwards to a point III on the great circle in question may be considered as being so drawn that in the enclosed spherical triangle I, II, III, the angle at III is a right angle. It follows that the hypotenuse I, II $= \tau$, the angle at I $= \pi$, and the opposite catheta II, III $= k$. For the calculation of k there is then available the equation:

$$\sin k = \sin \tau \sin \pi.$$

It will be understood that the various values of k arising from the successive refractions may be computed by substituting for the original values of π and τ those occurring after the first, second, &c., refractions, as determined by the scheme of computation given in the text.

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